FOURIER ANALYSIS ϕv. 1

① $\lim_{n\to\infty} z_n = z \implies \lim_{n\to\infty} \dfrac{z_1 + z_2 + \cdots + z_n}{n} = z$

② $$\sum_{n=-N}^{N} e^{int} = \frac{\sin[(N+\frac{1}{2})t]}{\sin(\frac{1}{2}t)}$$ Show this!

<u>Remark:</u> $D_N(t) = \dfrac{1}{2\pi}\sum_{-N}^{N} e^{int}$ is the Dirichlet Kernel.

③ The FEJÉR KERNEL is defined as

$$F_N(t) = \frac{D_0(t) + D_1(t) + D_2(t) + \cdots + D_N(t)}{N+1}$$

Arithmetic mean, Cesàro sum.

Show that

$$2\pi F_N(t) = \frac{1}{N+1}\left[\frac{\sin\left(\frac{N+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)}\right]^2 .$$

④ The "functions" $\varphi_1, \varphi_2, \varphi_3, \ldots$ are orthonormal in the Hilbert space $\mathcal{H}$, i.e.,

$$\langle \varphi_k, \varphi_j \rangle = \delta_{jk}$$

Show that $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ is linearly independent.

⑤ We <u>know</u> that the trigonometric system $e^{inx}/\sqrt{2\pi}$, $n = 0, \pm 1, \pm 2, \ldots$ is a basis in $L^2(-\pi,\pi)$. Show that the system

$$\frac{\sin(nx)}{\sqrt{\frac{\pi}{2}}} \qquad (n = 1, 2, 3, \ldots)$$

is a basis in the space $L^2(0, \pi)$.

(6) Assume that $f_n \longrightarrow f$ in $L^2(-\pi, \pi)$, i.e.,

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f(x) - f_n(x)|^2 \, dx = 0.$$

Prove that $\lim_{n \to \infty} c_k(f_n) = c_k(f)$, $k = 1, 2, 3, \ldots$,

for the Fourier coefficients

$$c_k(f_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f_n(x) \, dx$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$1 + q + q^2 + \cdots + q^{N-1} = \frac{1 - q^N}{1 - q} \qquad (q \neq 1)$$

$$|e^{i\theta}| = 1 \quad (\theta = \text{a real number})$$

$\lim_{n \to \infty} z_n = z \overset{\text{DEF.}}{\iff}$ Given $\varepsilon > 0$, there is an index n_ε such that

$(z \neq \infty)$ $\qquad |z_n - z| < \varepsilon$ when $n > n_\varepsilon$.