TEMPERED DISTRIBUTIONS
OR $\mathcal{S}'(\mathbb{R})$.

If $f \in L^1(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$,

then
$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle,$$
Change of Hats.

as an easy calculation shows. This suggests that the Fourier transform $\hat{T}$ of the associated distribution

$$T(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$$

should be defined as

$$\hat{T}(\phi) = \int_{-\infty}^{\infty} \hat{f}(x)\phi(x)\,dx = T(\hat{\phi}).$$

Thus $\hat{T}(\phi) = T(\hat{\phi})$.

OBSTACLE: ϕ and $\hat{\phi}$ cannot both belong to $C_0^\infty(\mathbb{R})$, except when $\phi = 0$.

To circumvent this obstacle we use a larger class of test functions, the SCHWARTZ class $\mathcal{S} = \mathcal{S}(\mathbb{R})$.

DEF.: $\phi \in \mathcal{S}(\mathbb{R})$ if $\phi \in C^\infty(\mathbb{R})$

and
$$\max_x \left| x^N \phi^{(n)}(x) \right| < \infty$$

for all $N = 1, 2, \ldots$, $n = 0, 1, 2, \ldots$

Such functions are smooth and rapidly decaying:

$$\lim_{x\to\pm\infty} |x|^N \left|\frac{d^n\phi(x)}{dx^n}\right| = 0$$

for all derivatives and powers.

Ex.: The functions

$$e^{-x^2},\quad x^7 e^{-x^2},\quad e^{-(1+x^2)^2},\quad \frac{1}{\cosh(x)}$$

belong to $\mathcal{S}$. The functions

$$e^{-|x|},\quad \sin(x),\quad \frac{1}{1+x^2}$$

do not belong to $\mathcal{S}$.

LEMMA $f, g \in \mathcal{S} \Rightarrow fg \in \mathcal{S},\ f*g \in \mathcal{S},$
$f' \in \mathcal{S},\ x f(x) \in \mathcal{S}.$

THEOREM $\phi \in \mathcal{S} \Longleftrightarrow \hat{\phi} \in \mathcal{S}$

This enables us to define the Fourier Transform $\hat{T}$ of a distribution T.

DEF. We say that $T : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ is a tempered distribution, if

(i) $T(\alpha\phi_1 + \beta\phi_2) = \alpha T(\phi_1) + \beta T(\phi_2)$ LINEARITY

(ii) $\phi_k \longrightarrow \phi$ in $\mathcal{S}(\mathbb{R}) \Rightarrow \lim_{k\to\infty} T(\phi_k) = T(\phi).$ "CONTINUITY"

Here the test functions converge in the sense

that
$$\max_x \left| |x|^N \frac{d^n}{dx^n}(\phi_k(x) - \phi(x)) \right| \longrightarrow 0,$$
as $k \longrightarrow \infty$, for each fixed $n = 0, 1, 2, \ldots;$ $N \geq 1$.

NOTATION $T \in \mathcal{S}' = \mathcal{S}'(\mathbb{R})$.

<u>DEF.</u>: $\hat{T}(\phi) = T(\hat{\phi})$ for all $\phi \in \mathcal{S}$.

<u>Ex.</u>: $T(\phi) = \phi(0)$ (Dirac's Delta)

$$\hat{T}(\phi) = T(\hat{\phi}) = \hat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x)\, dx$$

Thus $\hat{T} = \frac{1}{\sqrt{2\pi}}$.
$$\boxed{\hat{\delta} = \frac{1}{\sqrt{2\pi}}}$$

<u>Ex.</u>: $I(\phi) = \int_{-\infty}^{\infty} \phi(x)\, dx$ (Identity $\mathbb{1}$)

$$\hat{I}(\phi) = I(\hat{\phi}) = \int_{-\infty}^{\infty} \hat{\phi}(\omega)\, d\omega = \lim_{N \to \infty} \int_{-N}^{N} \hat{\phi}(\omega)\, d\omega$$

$$= \lim_{N \to \infty} \int_{-N}^{N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} \phi(x)\, dx\, d\omega$$

$$= \lim_{N \to \infty} \int_{-\infty}^{\infty} \phi(x) \underbrace{\int_{-N}^{N} \frac{e^{-i\omega x}}{\sqrt{2\pi}}\, d\omega}_{\text{Integrate!}}\, dx = \lim_{N \to \infty} \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) \frac{\sin(Nx)}{x}\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \phi(0)\pi.$$ To see this, write

$$\phi(x) = \phi(0) + x\left(\phi'(0) + \cdots\right)$$
$$= \phi(0) + x\, f(x).$$

$$\int_{-\infty}^{\infty} \frac{\sin(Nx)}{x}\, dx = \pi. \quad (N > 0)$$

$$\int_{-\infty}^{\infty} \phi(x) \frac{\sin(Nx)}{x}\, dx = \phi(0) \int_{-\infty}^{\infty} \frac{\sin(Nx)}{x}\, dx$$
$$+ \underbrace{\int_{-\infty}^{\infty} f(x) \sin(Nx)\, dx}_{\to 0 \text{ by the Riemann-Lebesgue lemma.}} \longrightarrow \phi(0)\cdot\pi + 0$$

FOURIER TRANSF. OF A CONSTANT C.

$$\boxed{\hat{C} = \sqrt{2\pi}\, C\, \delta}$$

$$\hat{1} = \sqrt{2\pi}\, \delta_0$$

OPERATIONS:

$$\left(\frac{d^n T}{dx^n}\right)(\phi) = (-1)^n\, T\left(\frac{d^n \phi}{dx^n}\right) \quad \text{DIFFERENTIATION}$$

REFLEXION $\quad T_\delta(\phi) = T(\phi_\delta) \qquad \phi_\delta(x) = \phi(-x)$

TRANSLATION $\quad (\tau_h T)(\phi) = T(\tau_{-h}\phi) \qquad (\tau_{-h}\phi)(x) = \phi(x+h)$

MINUS SIGN

$$\hat{T}(\phi) = T(\hat{\phi}) \quad \text{FOURIER TRANSFORM}$$

$$(mT)(\phi) = T(m\phi), \quad \text{where } m \in C^\infty(\mathbb{R})$$

We also write $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$,

$$\left\langle \frac{d^n T}{dx^n}, \phi \right\rangle = \left\langle T, \frac{d^n \phi}{dx^n} \right\rangle$$

and so on

$$\begin{cases} (\mathcal{F}\phi)(\omega) = \hat{\phi}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-i\omega x}\,\phi(x)\,dx \\ (\bar{\mathcal{F}}\phi)(\omega) = \check{\phi}(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{+i\omega x}\,\phi(x)\,dx \end{cases}$$

$$= (\mathcal{F}\phi_\delta)(\omega)$$

$$\phi = \bar{\mathcal{F}}\hat{\phi}, \quad \hat{\phi} = \mathcal{F}\phi \qquad \boxed{\underset{\text{INVERSE}}{\mathcal{F}^{-1}} = \bar{\mathcal{F}}}$$

We define

$$\langle \mathcal{F}T, \phi\rangle = \langle T, \mathcal{F}\phi\rangle$$

$$\langle \bar{\mathcal{F}}T, \phi\rangle = \langle T, \bar{\mathcal{F}}\phi\rangle = \langle T, \widehat{\phi_\delta}\rangle$$

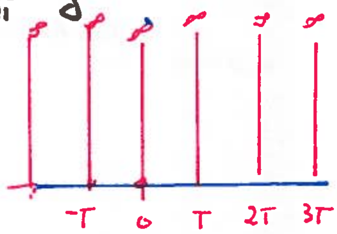
They are inverses

$$\bar{\mathcal{F}}\mathcal{F} = \text{Identity} = \mathcal{F}\bar{\mathcal{F}}$$

$$\boxed{\bar{\mathcal{F}}\mathcal{F}T = T = \mathcal{F}\bar{\mathcal{F}}T}$$

for distributions.

$$\begin{cases} \widehat{\dfrac{d^k T}{dx^k}} = (i\omega)^k\,\hat{T}. \\ \widehat{e^{iax}T} = \tau_a\hat{T}. \\ \hat{\delta} = \dfrac{1}{\sqrt{2\pi}}. \\ \hat{\delta}_a = \dfrac{1}{\sqrt{2\pi}}e^{-ia\omega}. \\ \widehat{\delta^{(k)}} = (i\omega)^k\dfrac{1}{\sqrt{2\pi}}. \end{cases} \qquad \begin{cases} \widehat{(-ix)^k T} = \dfrac{d^k\hat{T}}{d\omega^k}. \\ \widehat{\tau_a T} = e^{-ia\omega}\,\hat{T}. \\ \widehat{e^{iax}} = \sqrt{2\pi}\,\delta_a. \\ \hat{1} = \sqrt{2\pi}\,\delta. \\ \widehat{x^n} = (-i)^{-n}\sqrt{2\pi}\,\delta^{(n)} \end{cases}$$



$$Ш_T(t) \equiv \sum \delta(t-nT) \qquad \widehat{Ш_T}(\omega) = \frac{\sqrt{2\pi}}{T}\,Ш_{\frac{2\pi}{T}}(\omega)$$

Dirac's Comb "Shah"