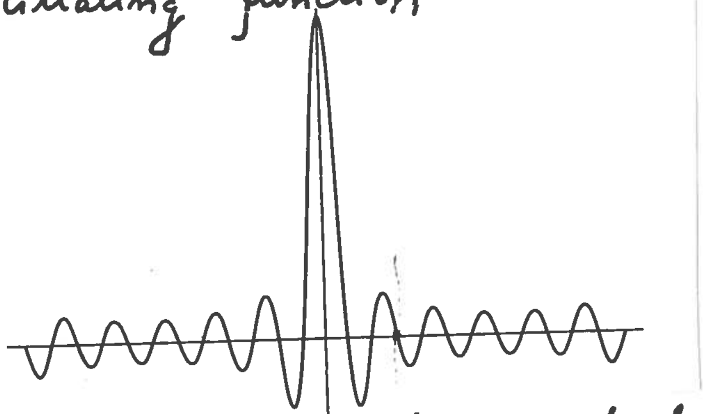
Peter Lindqvist 2001

POINTWISE CONVERGENCE
THE DIRICHLET AND FEJÉR KERNELS

The Dirichlet Kernel

$$2\pi D_N(t) = \sum_{n=-N}^{+N} e^{int} = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

is an oscillating function



The numbers of waves increases.

$$D_N(0) = \frac{2N+1}{2\pi}$$

focusing its behaviour at the point $t = 0$. Indeed,

$$\int_{-\pi}^{\pi} D_N(t)\,dt = 1, \quad \lim_{N\to\infty} \int_{-\delta}^{\delta} D_N(t)\,dt = 1$$

Effects of cancellation lead to

$$\lim_{N\to\infty} \int_{\delta}^{\pi} D_N(t)\,dt = 0. \qquad (0 < \delta \le \pi)$$

For smooth functions

$$\boxed{\int_{-\pi}^{\pi} \varphi(t) D_N(t)\,dt \longrightarrow \varphi(0)}$$

as $N \to \infty$. The Dirichlet Kernel appears in the important formula

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \int_{-\pi}^{\pi} f(x+t) D_N(t)\, dt$$

for the N^{th} partial sum of the Fourier series of the function $f \in L^1(-\pi,\pi)$ with period 2π. We also have

$$\boxed{f(x) - S_N(x) = -\int_{-\pi}^{\pi} \big(f(x+t) - f(x)\big) D_N(t)\, dt}$$

This formula can be used to produce the following theorem about the pointwise convergence

THEOREM Suppose that $\int_{-\pi}^{\pi} |f(t)|\, dt < \infty$. Let $-\pi < x < \pi$. If the derivative $f'(x)$ exists at the point x, then

$$f(x) = \lim_{N \to \infty} S_N(x).$$

In other words, the Fourier series converges to the value $f(x)$ at the point x.

The difficulty about the Dirichlet Kernel is that

$$\int_{-\pi}^{\pi} |D_N(t)|\, dt \approx \frac{4}{\pi^2} \ln N \longrightarrow \infty,$$

as $N \to \infty$, although $\int_{-\pi}^{\pi} D_N(t)\, dt = 1$. In contrast, the Fejér Kernel is a positive one.

The FEJÉR KERNEL is defined as

$$F_N(t) = \frac{D_0(t) + D_1(t) + D_2(t) + \cdots + D_N(t)}{N+1}$$

"Arithmetic Mean"

A calculation yields

$$2\pi F_N(t) = \frac{1}{N+1}\left[\frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}}\right]^2$$

We can read off

① $F_N(t) \geq 0$

② $F_N(0) = \dfrac{N+1}{2\pi}$

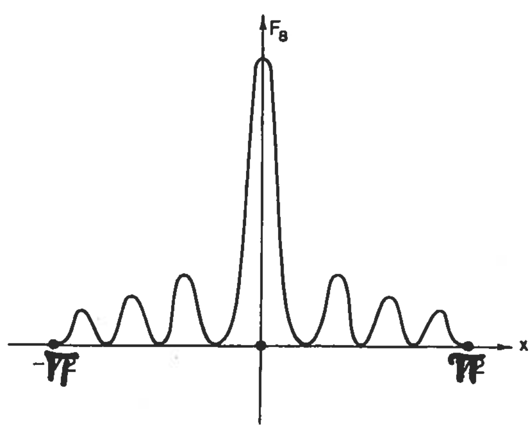
③ $\displaystyle\int_{-\pi}^{\pi} F_N(t)\,dt = 1, \qquad \int_{-\pi}^{\pi} |F_N(t)|\,dt = 1$

④ $F_N(t) = F_N(-t)$

⑤ If $0 < \delta \leq |t| < \pi$, then

$$0 \leq F_N(t) \leq \frac{1}{2\pi(N+1)\sin^2\left(\frac{\delta}{2}\right)} \longrightarrow 0$$

as $N \to \infty$.



The *Fejér Kernel* appears in the representation formula

$$\sigma_N(x) = \frac{1 + S_1(x) + \cdots + S_N(x)}{N+1} = \int_{-\pi}^{\pi} f(t+x) F_N(t)\,dt$$

for the so-called *Cesàro sums* of the Fourier series. As before,

$$S_N(x) = \int_{-\pi}^{\pi} f(x+t) D_N(t)\,dt.$$

THEOREM *Suppose that* f *is continuous in the interval* $[-\pi, \pi]$. Then

$$f(x) = \lim_{N\to\infty} \sigma_N(x),$$

when $|x| < \pi$.

Proof: Given $\varepsilon > 0$, there is a (small) δ such that

$$|f(x+t) - f(x)| < \varepsilon$$

when $|t| < \varepsilon$, because $\lim_{t\to 0} f(x+t) = f(x)$ by the assumption. Now

$$\sigma_N(x) - f(x) = \int_{-\pi}^{\pi} \big(f(x+t) - f(x)\big) F_N(t)\,dt$$

By the triangle inequality for integrals,

$$|\sigma_N(x) - f(x)| \leq \int_{-\pi}^{\pi} |f(x+t) - f(x)| F_N(t)\,dt$$

$$= \int_{|t|<\delta} \cdots\, dt + \int_{|t|\geq\delta} \cdots\, dt \quad \leq$$

$$\varepsilon \int_{|t|<\delta} F_N(t)\,dt + \int_{|t|>\delta} |f(x+t) - f(x)|\, F_N(t)\,dt$$

$$\underset{\substack{\uparrow \\ F_N \geq 0}}{\leq} \varepsilon \cdot 1 + \underbrace{\frac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}} \cdot 2\int_{-\pi}^{\pi} |f(t)|\,dt}_{\longrightarrow 0 \text{ as } N \to \infty}$$

$< \varepsilon + \varepsilon = 2\varepsilon$, when $N > N_\varepsilon$ (an index).

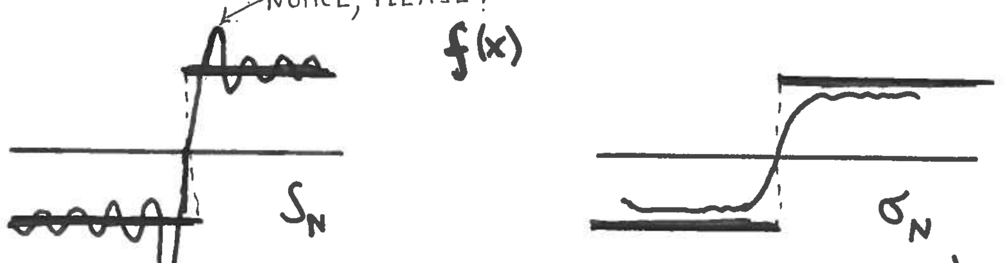
Since $\varepsilon > 0$ was arbitrary, this proves the theorem. ∎

<u>Remark</u> Notice that

$$|\sigma_N(x)| \leq \int |f(x+t)|\, F_N(t)\,dt$$

$$\leq \max|f| \cdot \int F_N(t)\,dt \leq \max_{[-\pi,\pi]} |f|$$

There is no Gibbs' phenomenon for the Cesàro sums σ_N.



FOURIER Rectangular pulse. CESÀRO

SOME REMARKS

1) The Fourier series of a continuous function can be divergent at a given point. However, the Cesàro sums converge to the correct value.

2) Kolmogoroff has constructed a function belonging to L^1 such that its Fourier series diverges to ∞ at each point in the interval!

3) Carleson's Theorem: If $f \in L^2(-\pi, \pi)$, then
1966

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

at almost every point $x \in (-\pi, \pi)$. In other words, the exceptional points form a set of length measure zero.

4) Cantor's Theorem for trigonometric series:
If

$$\sum_{-\infty}^{\infty} c_n e^{inx} = 0$$

for all x, then $c_n = 0$, $n = 0, \pm 1, \pm 2, \ldots$
(Nothing was assumed about the coefficients c_n.)

5) The convergence theorems about

$$\frac{f(x+0) + f(x-0)}{2}$$

hold for the class of functions of bounded variation.