Peter Lindqvist 2001

LEBESGUE'S MEASURE AND INTEGRAL

Ca 1902

- More measurable set
- More integrable functions
- Good limit theorems
- The Riemann integral is a special case
- L^2 is a complete space (Riesz-Fischer's Theorem)
- Easy to handle

OUTER MEASURE

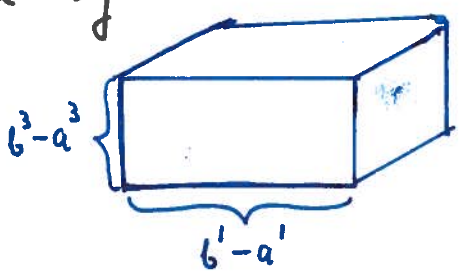
The outer measure of a set A in $\mathbb{R}$ is defined as the infimum of all the sums

$$\sum_{j=1}^{\infty}(b_j - a_j) \quad \text{where} \quad A \subset \bigcup_{j=1}^{\infty}(a_j, b_j).$$

INFINITELY MANY MUST BE ALLOWED!

The notation means that any $x \in A$ belongs to some interval (a_j, b_j). The outer measure is denoted $\text{mes}^*(A)$. Always $0 \le \text{mes}\, A^* \le \infty$. (In several dimensions, the intervals are replaced by

$$(a_j^1, b_j^1) \times (a_j^2, b_j^2) \times \cdots \times (a_j^n, b_j^n)$$



and the lengths by

$$(b_j^1 - a_j^1)(b_j^2 - a_j^2)\cdots(b_j^n - a_j^n).)$$

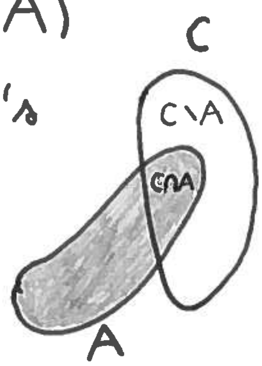
Unfortunately, the situation

$$\text{mes}^*(A \cup B) < \text{mes}^* A + \text{mes}^* B$$

!

is possible, even if $A \cap B = \phi$. To avoid such phenomena, the collection of measurable sets is singled out. We say that A is measurable in the sense of Lebesgue, if

$$\mathrm{mes}^*(C) = \mathrm{mes}^*(C \cap A) + \mathrm{mes}^*(C \setminus A)$$



for all sets C. (This is Carathéodory's Criterion for measurability.) If A is measurable, we write

$$\mathrm{mes}\, A = \mathrm{mes}^* A.$$

The measurable sets have the following properties:

(i) ϕ and $\mathbb{R}$ are measurable.

(ii) If $A_1, A_2, A_3, \ldots$ are measurable, so are

$$\bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \bigcap_{j=1}^{\infty} A_j .$$

(iii) If A is measurable, so is the complement $\mathbb{R} \setminus A$.

(iv) If $A_1 \subset A_2 \subset A_3 \subset \cdots$ are measurable, then

$$\mathrm{mes}\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \lim_{j \to \infty} \mathrm{mes}\, A_j .$$

(v) For disjoint measurable sets we have

$$\mathrm{mes}\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mathrm{mes}\, A_j , \qquad A_i \underset{i \neq j}{\cap} A_j = \phi .$$

The important additivity property (v) has to be replaced by subadditivity

$$\mathrm{mes}^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mathrm{mes}^* A_j$$

if non-measurable sets are involved. There are plenty of measurable sets in $\mathbb{R}$ (or in $\mathbb{R}^n$).

Theorem: If $\text{mes}^* A = 0$, then A is measurable.

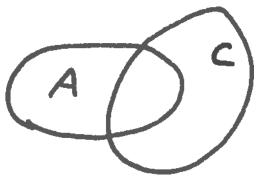
Proof: Always $\text{mes}^*(C) \leq \text{mes}^*(C \cap A) + \text{mes}^*(C \setminus A)$ by subadditivity, because

$$C = (C \cap A) \cup (C \setminus A).$$

We have to show that

$$m^*(C) \geq \text{mes}^*(C \cap A) + \text{mes}^*(C \setminus A)$$

for all "test-sets" C. Now



$$\text{mes}^*(C \cap A) \leq \text{mes}^*(A) = 0,$$
$$\text{mes}^*(C \setminus A) \leq \text{mes}^*(C),$$

since $C \cap A \subset A$ and $C \setminus A \subset C$. ∎

Example Let Q denote the set consisting of all the rational points $\frac{m}{n}$; $n = \pm 1, \pm 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$ They can be numbered like

$$q_1, q_2, q_3, \ldots, q_n, \ldots,$$

which is well-known. Let $\varepsilon > 0$. Obviously,

$q_j \in \left(q_j - \frac{\varepsilon}{2^{j+1}}, q_j + \frac{\varepsilon}{2^{j+1}} \right)$, an interval of length $\varepsilon / 2^j$. Thus

$$\text{mes}^* Q \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

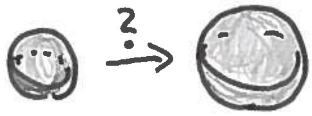
The rational points have "length" zero.

and, since $\varepsilon > 0$ was arbitrary, $\text{mes}^* Q = 0$.

In fact, any <u>countable</u> (= denumerable) set has measure zero. (There exist sets of measure zero that, however, are not countable.)

> **<u>Theorem</u>:** <u>In</u> $\mathbb{R}^n$ <u>every</u> <u>open</u> <u>set</u> <u>is</u> <u>measurable</u>. <u>So</u> <u>is</u> <u>every</u> <u>closed</u> <u>set</u>.

The Axiom of Choice is required to construct non-measurable sets. Strange phenomena occur. For instance, in $\mathbb{R}^3$ a ball of radius 1 can be divided into 5 congruent pieces so that these same pieces can be put together to form a ball of radius 2. Needless to say, measurable pieces are out of the question for this doubling.



THE INTEGRAL OF A BOUNDED FUNCTION

The rich supply of measurable sets is needed to guarantee the success of the Lebesgue integral! Let us define the integral of

$$f : A \longrightarrow \mathbb{R}$$

where 1) $\operatorname{mes} A < \infty$ and 2)

$$-\infty < g \leq f(x) \leq G < \infty, \quad \text{when } x \in A.$$

In other words, f is a <u>bounded function</u> and A, the set of integration, is of <u>finite measure</u>. Consider all subdivisions of A

into a finite number of disjoint measurable parts E_j. Let D symbolize the division

$$A = \bigcup_{j=1}^{p} E_j \,, \quad E_j \cap E_k = \phi \quad (j \neq k).$$

Form the sums

UPPER SUM

$$S_D = \sum_{j=1}^{p} G_j \operatorname{mes} E_j \,, \quad G_j = \sup_{x \in E_j} f(x)$$

LOWER SUM

$$s_D = \sum_{j=1}^{p} g_j \operatorname{mes} E_j \,, \quad g_j = \inf_{x \in E_j} f(x)$$

In the case of Riemann's integral, one restricts oneself to the case where all the E_j's are "intervals". – Now there are many more subdivisions. Continuing the construction, we observe that

$$g \operatorname{mes} A \leq s_{D_1} \leq S_{D_2} \leq G \operatorname{mes} A$$

for any subdivisions. Define

$$\overline{I} = \inf_D S_D \,, \quad \underline{I} = \sup_D s_D$$

and notice that

$$g \operatorname{mes} A \leq \boxed{\underline{I} \leq \overline{I}} \leq G \operatorname{mes} A.$$

DEFINITION <u>If $\underline{I} = \overline{I}$, we say that the bounded function f is integrable (in the sense of Lebesgue) over the set</u> A.

The ordinary notation is used:

$$\int_A f(x)\,dx, \quad \int_A f, \quad \int \cdots \int_A f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n$$

and so on. Notice the following comparison with the Riemann integral:

$$\underline{I}^{\text{Riemann}} \le \underline{I}^{\text{Lebesgue}} \le \overline{I}^{\text{Lebesgue}} \le \overline{I}^{\text{Riemann}}$$

A Riemann sum is always a Lebesgue sum. There are more Lebesgue sums.

Thus: $\underline{I}^{\text{Riemann}} = \overline{I}^{\text{Riemann}}$ (Riemann integrability)

$\Longrightarrow \underline{I}^{\text{Lebesgue}} = \overline{I}^{\text{Lebesgue}}$

<u>Theorem</u>: If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded and Riemann integrable, then f is Lebesgue integrable. The integrals coincide. $(b-a < \infty.)$

<u>Example</u> Dirichlet's discontinuous function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \text{ is irrational} \end{cases} \qquad A = [0,2].$$

If A is divided into a finite number of subintervals, we obviously have $s = 0$, $S = 2$. Hence

$$\underline{I}^{\text{Riemann}} = 0 < 2 = \overline{I}^{\text{Riemann}}.$$

The function is not Riemann integrable over $[0,2]$.

For Lebesgue's integral, the subdivision

$$E_1 = \mathbb{Q} \cap [0,2], \qquad E_2 = [0,2] \setminus \mathbb{Q}$$

is allowed. Then

$$s = 1 \operatorname{mes} E_1 + 0 \operatorname{mes} E_2 = 1\cdot 0 + 0 \cdot 2 = 0$$

$$S = \quad \text{———} \quad '' \quad \text{———}$$

so that

$$0 = s \le \underline{I}^{\text{Leb.}} \le \overline{I}^{\text{Leb}} \le S = 0.$$

Hence

$$\int_0^2 f(x)\,dx = 0 \quad (\text{Lebesgue}).$$

(Notice, $0 \le 0 \le 0 \le 2$, as it should be.)

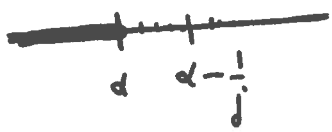
MEASURABLE FUNCTIONS

Lebesgue's theory deals only with <u>measurable functions</u>. We say that the function $f : A \longrightarrow [-\infty, \infty]$ is measurable, if each of the sets

$$\{x \mid f(x) > \alpha\}$$

is measurable. Then also the sets $\{x \mid f(x) \ge \alpha\}$, $\{x \mid f(x) < \beta\}$, $\{x \mid f(x) \le \beta\}$, and

$$\{x \mid \alpha < f(x) \le \beta\}$$



will be measurable. For example,

$$\{x \mid f(x) \ge \alpha\} = \bigcap_{j=1}^{\infty} \{x \mid f(x) > \alpha - \tfrac{1}{j}\},$$

and the intersection of measurable sets is measurable.

If f is allowed to take the values $+\infty$ and $-\infty$, one has to include that the sets $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are measurable.

THEOREM (Lebesgue) Suppose that $f: A \longrightarrow \mathbb{R}$ is a bounded function and that $\operatorname{mes} A < \infty$. Then the Lebesgue integral

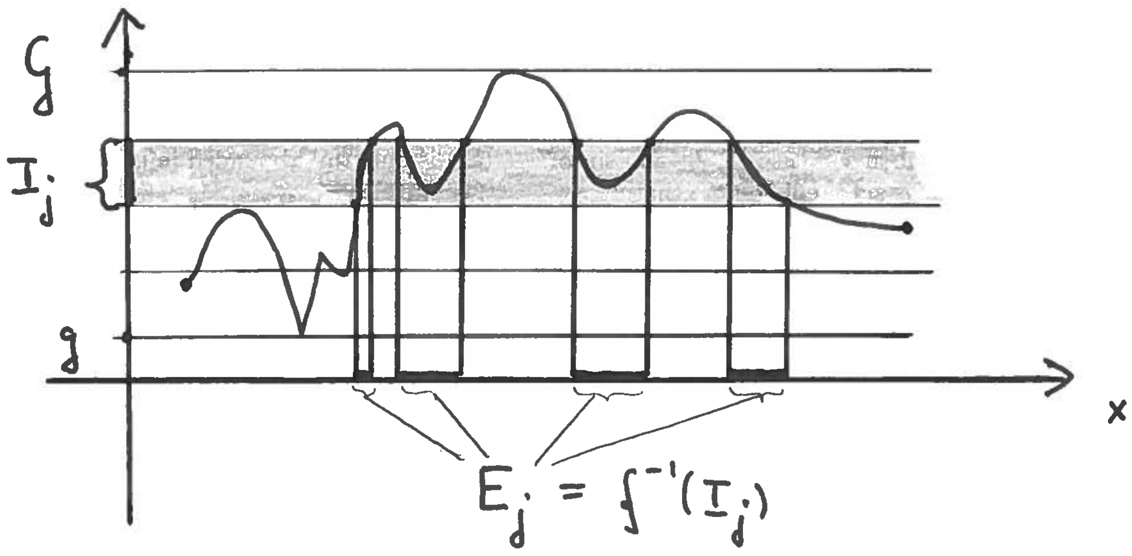
$$\int_A f(x)\,dx$$

exists, if and only if, f is a measurable function.

<u>Proof</u> of "$\Leftarrow$". Suppose that f is measurable. Denote $g = \inf_{x\in A} f(x)$, $G = \sup_{x\in A} f(x)$. Divide the interval $[g, G]$ into m equal subintervals

$$I_j = \left[g + \frac{j-1}{m}(G-g),\ g + \frac{j}{m}(G-g)\right),$$

$j = 1, 2, 3, \dots, m$. The right endpoint G is included in the last interval I_m.



This division of "the y-axis" induces a subdivision of A into disjoint sets

$$E_j = \{x \in A \mid f(x) \in I_j\}$$

$$= \{x \in A \mid g + \tfrac{j-1}{m}(G-g) \leq f(x) < g + \tfrac{j}{m}(G-g)\}.$$

E_j is measurable, because of the assumption that f is measurable. We have the division

$$A = \bigcup_{j=1}^{m} E_j , \quad E_j \cap E_k = \phi \quad (k \neq j) .$$

Now, given $\varepsilon > 0$,

$$S_D - s_D \leq \sum_{j=1}^{m} \frac{G-g}{m} \operatorname{mes}(E_j)$$

$$= \frac{G-g}{m} \operatorname{mes}(A) \quad < \varepsilon$$

when m is large enough. This clearly implies that $\underline{I} = \overline{I}$ (recall $s_D \leq \underline{I} \leq \overline{I} \leq S_D$ so that $\overline{I} - \underline{I} \leq S_D - s_D < \varepsilon$). Thus the Lebesgue integral of f over A exists. □

This was the case of bounded functions with sets of integration of finite measure. Next we consider POSITIVE functions. We have seen that the functions have to be measurable.

Let $A \subset \mathbb{R}$ (or $\mathbb{R}^n$) be a measurable set (mes $A = \infty$ is now allowed) and suppose that

$$f : A \longrightarrow [0, \infty]$$

is a measurable function. It is essential here that $f(x) \geq 0$. <u>We define</u>

$$\int_A f(x)\,dx = \sup_D s_D$$

the supremum being taken over all measurable subdivisions

$$A = \bigcup_{j=1}^{p} E_j, \quad E_j \cap E_k = \phi \ (j \neq k).$$

Notice that the upper sums S_D are not evoked! (In fact, the assumption that f be a measurable function compensates for this absence.) The result is

$$0 \leqq \int_A f(x)\,dx \leqq +\infty.$$

(The definition can be rewritten in terms of integrals of step-functions ("simple" functions)

$$s(x) = \sum_{j=1}^{p} c_j \mathbb{1}_{E_j}(x)$$

corresponding to subdivisions of A. The step-function takes the constant value c_j in E_j. Now

$$\int_A f(x)\,dx = \sup_{s \le f} \int_A s(x)\,dx,$$

where

$$\int_A s(x)\,dx = \sum_{j=1}^{p} c_j \operatorname{mes} E_j .$$

Notice that Dirichlets discontinuous function will do as a step-function.)

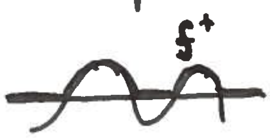
GENERAL CASE $f^+ - f^-$

This was the integral of positive functions. It obeys many natural rules like

$$\int_A (a f(x) + b g(x))\,dx = a\int_A f(x)\,dx + b\int_A g(x)\,dx .$$

The general case goes via the decomposition

$$f = f^+ - f^-$$



$$f^+(x) = \max\{f(x), 0\}, \qquad f^-(x) = -\min\{f(x), 0\}$$

If f is measurable, so are f^+ and f^-. Now $f^+(x) \ge 0$ and $f^-(x) \ge 0$, so that <u>the integrals</u>

$$\int_A f^+(x)\,dx, \qquad \int_A f^-(x)\,dx$$

<u>are defined as above</u>. We define

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx,$$

the situation $\infty - \infty$ being excluded.

DEFINITION <u>Let</u> $f : A \longrightarrow [-\infty, \infty]$ <u>be a measurable function. We say that</u> f <u>is summable (integrable) over</u> A <u>in the sense of Lebesgue</u>, <u>if</u>

$$\int_A f^+(x)\,dx < \infty, \quad \int_A f^-(x)\,dx < \infty$$

<u>We define</u>

$$\int_A f(x)\,dx = \int_A f^+(x)\,dx - \int_A f^-(x)\,dx.$$

Notation: $f \in L^1(A)$.

Notice that this is an <u>absolutely</u> convergent integral:

$$\int_A |f(x)|\,dx = \int_A f^+(x)\,dx + \int_A f^-(x)\,dx,$$

since $|f| = f^+ + f^-$. The <u>improper</u> Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}, \quad \int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$$

is not of this type. In this example

$$\int_A f^+ - \int_A f^- = \infty - \infty = ??$$

and so the procedure above does not produce $\pi/2$. However, the improper Riemann integral does not come directly from sums corresponding to a subdivision of the whole half-axis. One has to go via the integrals over the <u>finite</u> intervals $[0, c]$. Indeed, one means

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{c\to\infty} \int_0^c \frac{\sin x}{x}\,dx.$$

Nothing hinders us from considering <u>improper</u> Lebesgue integrals like

$$\lim_{c\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} f(x)e^{ixt}\,dx$$

The integrals over $[-c, c]$ are "ordinary" Lebesgue integrals.

Properties of the integral

① $$\int_A (af(x) + bg(x))\,dx = a\int_A f(x)\,dx + b\int_A g(x)\,dx$$

LINEARITY $\qquad f, g \in L^1(A)$

② $$\left| \int_A f(x)\,dx \right| \le \int_A |f(x)|\,dx, \qquad f \in L^1(A)$$

TRIANGLE INEQUALITY.

(3) $$\int_{\bigcup_{j=1}^{\infty} E_j} f(x)\,dx = \sum_{j=1}^{\infty} \int_{E_j} f(x)\,dx$$ (Either $f \geq 0$ or $f \in L^1$)

COUNTABLE ADDITIVITY

Disjoint measurable sets

(4) Let $f \in L^1(A)$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \int_E f(x)\,dx \right| < \varepsilon$$

ABSOLUTE CONTINUITY

whenever $\operatorname{mes}(E) < \delta$.

ADDENDUM about <u>measurable</u> <u>functions</u>. They form a collection, closed under many operations. If f and g are measurable, so are $f+g$, fg, $1/f$ and $|f|$. More generally, $|f|^p$ is measurable. If the functions $f_1, f_2, f_3, \dots$ are measurable, so are the limits

$$\liminf f_k, \quad \limsup f_k, \quad \lim f_k$$

if they exist. Also $\inf\{f_1, f_2, \dots\}$ and $\sup\{f_1, f_2, \dots\}$ are measurable. CAUTION The composed function $f \circ g$ can fail to be measurable. (The assumptions about f are crucial.)

In particular, $f'(x)$ is measurable if $f(x)$ is, since

$$f'(x) = \lim_{k \to \infty} \frac{f(x + \frac{1}{k}) - f(x)}{\frac{1}{k}},$$ a limit of measurable functions.

The differentiability is, of course, assumed.

Convergence results

LEBESGUE'S MONOTONE CONVERGENCE THEOREM: Suppose that each f_k is measurable and that

$$0 \le f_1(x) \le f_2(x) \le \cdots$$

when $x \in A$. Then

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = \int_A \left(\lim_{k\to\infty} f_k(x)\right) dx.$$

Notice that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

exists as a real number or is $+\infty$, because of the monotonicity.

COROLLARY Suppose that each u_k is a measurable function and that $u_k(x) \ge 0$, when $x \in A$. Then

$$\int_A \left(\sum_{k=1}^{\infty} u_k(x)\right) dx = \sum_{k=1}^{\infty} \int_A u_k(x)\,dx.$$

The series with positive terms can be integrated termwise, because the partial sums

$$u_1(x) + u_2(x) + \cdots + u_k(x)$$

are increasing with k.

<u>Ex.:</u> $$\lim_{n\to\infty} \int_0^{\infty} \left(1+\frac{x}{n}\right)^n e^{-x}\,dx \overset{\text{MON.}}{\underset{\text{CONV. THM.}}{=}} \int_0^{\infty} e^{x} e^{-x}\,dx = \infty.$$

LEBESGUE'S DOMINATED CONVERGENCE THEOREM: Suppose that $f(x) = \lim f_k(x)$, when $x \in A$ and that the measurable functions f_k satisfy

$$|f_k(x)| \leq g(x), \text{ when } x \in A, \quad (k = 1, 2, 3, \ldots)$$

for some function $g \in L^1(A)$. Then

$$\lim_{k\to\infty} \int_A f_k(x)\,dx = \int_A f(x)\,dx.$$

IT IS ESSENTIAL THAT $\int_A g(x)dx < \infty$

<u>Remark</u>: The special case, when $\text{mes}A < \infty$ and $|f_k(x)| \leq M$ (a constant), is called Lebesgue's <u>Bounded Convergence Theorem</u>. (Then $g(x) \equiv M$ will do as majorant.)

<u>Example</u> $\lim_{n\to\infty} \int_0^\infty e^{-x^n} dx = ?$

$$\lim_{n\to\infty} e^{-(x^n)} = \begin{cases} 0, & \text{when } x > 1 \\ e^{-1}, & \text{when } x = 1 \\ 1, & \text{when } 0 \leq x < 1 \end{cases}$$

$$\left|e^{-x^n}\right| \leq \left\{\begin{matrix} e^{-x}, & x \geq 1 \\ 1, & 0 \leq x \leq 1 \end{matrix}\right\} = g(x)$$

Since $\int^\infty g(x)dx = 1 + \frac{1}{e} < \infty$, we can use the dominated convergence theorem to pass the limit under the integral. <u>Answer</u>: $? = 1$.

FATOU'S LEMMA Suppose that f_k is measurable and $f_k \geq 0$. Then

$$\int_A (\liminf f_k(x))\,dx \leq \liminf_{k\to\infty} \int_A f_k(x)\,dx$$

"CONTINUITY IN THE MEAN" If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then

$$\lim_{h\to 0} \int |f(x+h) - f(x)|^p\,dx = 0.$$

THE RIESZ-FISCHER THEOREM The L^p-space $(p \geq 1)$ is complete.

COROLLARY If $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$, then the series

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

converges in the $L^2(-\pi,\pi)$-norm to a function in $L^2(-\pi,\pi)$.

APPROXIMATION

A function in L^p can happen to be discontinuous at every point. However, it can be approximated by smooth functions. The case $L^2(-\pi,\pi)$ is very appealing. If $f \in L^2(-\pi,\pi)$, then

$$\lim_{+m,+n\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \sum_{-m}^{n} c_k e^{ikx} \right|^2 dx = 0$$

where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$ is the Fourier coefficient. Notice that the approximating functions

$$\sum_{-m}^{n} c_k e^{ikx} \qquad (m, n \text{ finite indices})$$

are trigonometric polynomials, having infinitely many derivatives.

LEBESGUE POINTS

Finally, let us consider the Lebesgue points. This advanced concept is relevant for the Harmonic Analysis. Let

$$B(x,r) = \{ y \in \mathbb{R}^n \mid |x-y| < r \}$$

denote the open ball with radius r, centered at x. In one dimension it reduces to the interval $(x-r, x+r)$ of length $2r$. Let

$$|B(x,r)| = \int_{B(x,r)} dx = \int_{B(0,r)} dx$$

denote the volume. It is clear that

$$\lim_{r \to 0+} \frac{1}{|B(0,r)|} \int_{B(x,r)} \varphi(y)\, dy = \varphi(x)$$

if φ is continuous. In one dimension we take the limit of the average value

$$\frac{1}{2r} \int_{x-r}^{x+r} \varphi(y) dy = \varphi(x) + \frac{1}{2r} \int_{x-r}^{x+r} \big(\varphi(y) - \varphi(x)\big) dy$$

as $r \to 0+$.

LEBESGUE'S DIFFERENTIATION THEOREM

Let $f \in L^1(\mathbb{R}^n)$. Then

$$(*) \qquad f(x) = \lim_{r \to 0+} \frac{1}{|B(0,r)|} \int_{|x-y|<r} f(y)\,dy$$

for a.e x.

a.e. = almost every. This means that those points x at which the above identity does not hold form <u>a</u> <u>set</u> <u>of</u> Lebesgue <u>measure</u> <u>zero</u>.

DEFINITION We say that x is <u>a</u> <u>Lebesgue</u> <u>point</u> of $f \in L^1(\mathbb{R}^n)$, if

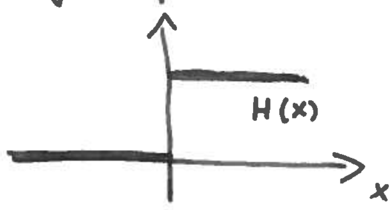
$$(**) \qquad \lim_{r \to 0+} \frac{1}{|B(0,r)|} \int_{B(x,r)} |f(y) - f(x)|\,dy$$

Notice that $(**) \Rightarrow (*)$.

THEOREM Almost every point is a Lebesgue point of $f \in L^1(\mathbb{R}^n)$.

<u>Ex.</u> The origin is <u>not</u> a Lebesgue point of the function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$



no matter how $H(0)$ is defined.

Ex. The Lebesgue points of Dirichlet's discontinuous function are (precisely) the irrational points.

The Lebesgue points appear in the following connexion in Harmonic Analysis.

THEOREM Let $f \in L^1(-\pi,\pi)$. Then

$$f(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) e^{ikx}$$

at every Lebesgue point of f. Thus the Cesàro means of the Fourier series converge a.e. to the correct value.

THEOREM Let $f \in L^1(\mathbb{R})$. Then the limits of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} e^{-\alpha|\omega|} d\omega \qquad (\alpha \to 0+) \quad \text{ABEL}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} e^{-\alpha\omega^2} d\omega \qquad (\alpha \to 0+) \quad \text{GAUSS}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \hat{f}(\omega) e^{i\omega t} \cdot \left(1 - \frac{|\omega|}{c}\right) d\omega \qquad (c \to \infty) \quad \text{FEJÉR-CESÀRO}$$

coincide with $f(t)$ at each Lebesgue point t. Thus we can invert the Fourier transform a.e..

Remark Kolmogoroff has exhibited a function $f \in L^1$ such that

$$\lim_{c\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-c}^{c} \hat{f}(\omega) e^{i\omega t} d\omega \equiv \infty \;!$$

Ibidem