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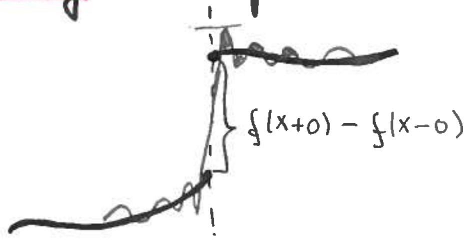
GIBBS' PHENOMENON

Near a jump discontinuity the partial sums S_N overshoot the jump

$$|f(x+0) - f(x-0)|$$



with about 8,9 %. There is a similar undershooting. To understand this, let us consider the example (pp. 85-87 in B&N) 6

$$f(t) = \begin{cases} \pi - t, & 0 < t \le \pi \\ 0, & t = 0 \\ -\pi - t, & -\pi \le t < 0 \end{cases}$$

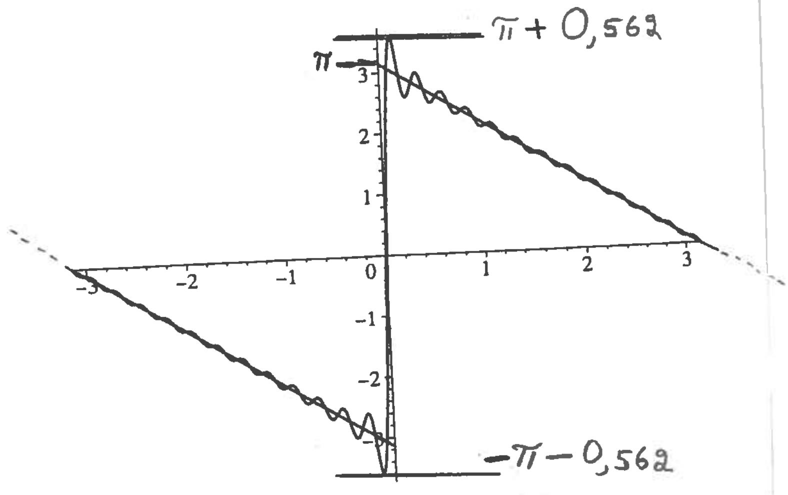
The Fourier coefficients are easily calculated:

$$a_n = 0, \quad b_n = \frac{2}{n}.$$

We have

$$f(t) = 2\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \qquad (-\pi \le t \le \pi)$$

at each point (notice that $f(0) = \dfrac{f(0+0)+f(0-0)}{2}$).



The first maximum, to the right of the origin, for the difference

$$g_N(t) = S_N(t) - f(t) = 2\sum_{n=1}^{N} \frac{\sin nt}{n} - (\pi - t),$$

where $0 < t \leq \pi$ occurres at a point where $g_N'(t) = 0$. We have

$$g_N'(t) = 1 + 2\sum_{n=1}^{N} \cos(nt) = \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t},$$

the first zero of which is

$$\theta_N = \frac{\pi}{N+\frac{1}{2}}. \qquad (\underline{\text{Remark}}: \ \theta_N \approx 0)$$

We calculate $g(\theta_N)$ through the formula

$$g_N(\theta_N) - g(0) = \int_0^{\theta_N} g_N'(t)\,dt$$

$$= \int_0^{\theta_N} \frac{\sin(N+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt = \int_0^{\pi} \frac{\sin\tau}{(N+\frac{1}{2})\sin\frac{\tau}{2(N+\frac{1}{2})}}\,d\tau$$

$$[\tau = (N+\tfrac{1}{2})t]$$

$$= 2\int_0^{\pi} \frac{\sin\tau}{\tau} \frac{d\tau}{\left(\dfrac{\sin\frac{\tau}{2(N+\frac{1}{2})}}{\frac{\tau}{2(N+\frac{1}{2})}}\right)} \xrightarrow[N\to\infty]{} 2\int_0^{\pi} \frac{\sin\tau}{\tau}\,d\tau$$

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Thus

$$\lim_{N\to\infty} g_N(\theta_N) = 2\int_0^{\pi} \frac{\sin\tau}{\tau} - \pi \approx 0{,}562$$

(NOT ZERO!)

This effect is not neglible!