The Fourier Transform of the distribution

Positive part.

$$\langle |\sin|_+ , \phi \rangle = \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \sin(x)\, \phi(x)\, dx$$

is by definition

$$\langle \widehat{|\sin|_+} , \phi \rangle = \langle |\sin|_+ , \hat{\phi} \rangle = \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \sin(\omega)\, \hat{\phi}(\omega)\, d\omega$$

$$= \sum_{-\infty}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \frac{e^{i\omega} - e^{-i\omega}}{2i} \int_{-\infty}^{\infty} e^{-i\omega x} \phi(x)\, dx\, d\omega \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) \underbrace{\left(\int_{2n\pi}^{(2n+1)\pi} \frac{e^{i\omega(1-x)} - e^{-i\omega(1+x)}}{2i}\, d\omega \right)}_{\frac{1}{\cancel{2}} e^{-2in\pi x}\left(1 + e^{-i\pi x}\right) \frac{\cancel{2}}{1-x^2} \text{ (by integration)}} dx \cdot \frac{1}{\sqrt{2\pi}}$$

Symmetric partial sums:

$$\frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{+N} \int_{-\infty}^{\infty} \frac{1 + e^{-i\pi x}}{1 - x^2}\, e^{-2in\pi x}\, \phi(x)\, dx = \cdots \int \sum_{-N}^{N}$$

Singularities at $x = \pm 1$ are removable!

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{2\pi D_N(2\pi x)}_{\longrightarrow \sum \delta(x-k)} \frac{1 + e^{-i\pi x}}{1 - x^2}\, \phi(x)\, dx$$

$$\boxed{2\pi D_N(t) = \sum_{n=-N}^{N} e^{int}}$$

DIRICHLET KERNEL $\longrightarrow \sum_{n=-\infty}^{\infty} \delta(x - 2k\pi)$

$$\longrightarrow \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{1 + e^{-i\pi k}}{1 - k^2}\, \phi(k)$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ 2\phi(0) - \frac{1}{2} i\pi\, \phi(1) + \frac{1}{2} i\pi\, \phi(-1) + \sum_{|k| \geq 2} \frac{1 + e^{-i\pi k}}{1 - k^2}\, \phi(k) \right\}$$