FOURIER ANALYSIS / week 13

(1) Mexican Hat. Verify

$$\begin{cases} \psi(t) = \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} (1-t^2)\, e^{-t^2/2} \\ \hat{\psi}(\omega) = \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}}\, \omega^2 e^{-\omega^2/2} \end{cases}$$

Can you figure out how the constant $\frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}}$ is "designed"?

(2) $\sum_{k=1}^{\infty} |a_k| < \infty \Rightarrow \prod_{k=1}^{\infty} (1+a_k)$ converges

What more can you say, if $a_k \geq 0$, $k = 1, 2, 3, \ldots$

(3) Let

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^2 \frac{1}{\sqrt{1 - \frac{2}{3} \sin^2\left(\frac{\omega}{2}\right)}}$$

Show that the translated functions $\phi(x-k)$, $k = 0, \pm 1, \pm 2, \ldots$ are orthonormal.

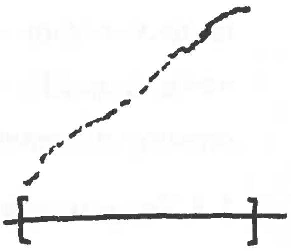
Remark: $$\sum_{k=-\infty}^{\infty} \frac{1}{(\omega + 2k\pi)^4} = \frac{3 - 2\sin^2\left(\frac{\omega}{2}\right)}{48 \sin^4\left(\frac{\omega}{2}\right)}$$

follows upon differentiating

$$\frac{1}{\sin^2 \frac{\omega}{2}} = \sum_{-\infty}^{\infty} \frac{1}{(\omega + 2k\pi)^2}$$

twice.

(4) Assume that $f : [0, 2\pi] \longrightarrow \mathbb{R}$ is monotone, say increasing. Prove that the Fourier coefficients



$$c_n = \frac{1}{2\pi} \int_0^{\pi} f(x) e^{-inx} \, dx$$

satisfy the inequality

$$|c_n| \le \frac{|f(2\pi - 0) - f(0+0)|}{4\,|n|} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

<u>Remark</u>: A monotone function may have infinitely many discontinuities, so-called jumps. The Riemann integral exists for bounded monotone functions.