Fourier Analysis 2015 / week 5.

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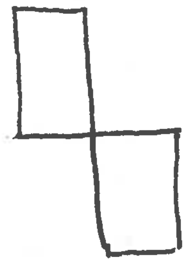
(1) Let $\overset{\text{signum}}{\operatorname{sgn}(x)} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$. The



RADEMACHER functions

$$r_n(x) = \operatorname{sgn}(\sin(2^n \pi x)), \quad n = 0, 1, 2, \ldots,$$

are orthonormal in $L^2(0,1)$. Prove that $\{r_0, r_1, r_2, r_3, \ldots\}$ is not a basis in (the Hilbert space) $L^2(0,1)$.

(2) (Describe how to) find the minimum

$$\min_P \int_{-1}^{1} (\sin(x) - P(x))^2 \, dx$$

taken among all polynomials $P(x)$ of degree $\leq 3$.

(3) Expand

$$f(x) = \begin{cases} \frac{1}{2}(\pi - 1)x, & 0 \leq x \leq 1 \\ \frac{1}{2}(\pi - x)1, & 1 \leq x \leq \pi \end{cases}$$

in a Fourier sine series. Then use the known formula

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2} \qquad (0 < x < 2\pi)$$

to conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n}\right)^2 = \frac{\pi - 1}{2}$$

④ Suppose that $f(x)$ is continuous in $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$. Assume that the finite limit

$$S(x_0) = \lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx} \quad (f\text{'s Fourier series})$$

exists at some point $x_0$, $-\pi \le x_0 \le \pi$. Is it possible that $S(x_0) \neq f(x_0)$ ?

⑤ Find the Fourier transform of the functions

$$f(x) = e^{-\alpha|x|}, \qquad g(x) = \begin{cases} e^{-\alpha|x|}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

where $\alpha > 0$.

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$$\mathcal{F}\{f\} = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

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A DEFINITION. A set $A \subset \mathbb{R}$ has measure zero, if given $\varepsilon > 0$, one can cover $A$ by intervals $I_j^\varepsilon = (a_j^\varepsilon, b_j^\varepsilon)$ so that

$$A \subset \bigcup_{j=1}^{\infty} I_j^\varepsilon, \qquad \sum_{j=1}^{\infty} (b_j^\varepsilon - a_j^\varepsilon) < \varepsilon$$

Denumerable covering

"length"