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THE DISTRIBUTION x^{-2}

DEF. $$\langle x^{-2}, \phi \rangle = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\phi(x)+\phi(-x)-2\phi(x)}{x^2}\,dx$$

This converges because near $x = 0$

$$\phi(x)+\phi(-x)-2\phi(x) = \phi''(0)x^2 + O(x^4)$$

Taylor's polynomial

The integrand is a bounded function.

LEMMA: $$\langle x^{-2}, \phi \rangle = \lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2}\,\phi(x)\,dx$$

Proof: Replacing x by $-x$, we see that

$$\int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2}\,\phi(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)}\big(\phi(x)+\phi(-x)\big)\,dx$$

For $\varepsilon \neq 0$

$$\int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2}\,dx = -\Big/_{-\infty}^{\infty} \frac{x}{x^2+\varepsilon^2} = -0-0 = 0$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2}\,\phi(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2-\varepsilon^2}{(x^2+\varepsilon^2)^2}\big[\phi(x)+\phi(-x)-2\phi(0)\big]\,dx$$

Now we may <u>safely</u> take the limit as $\varepsilon \to 0$. $\square$

Some other ways of representing x^{-2} as a distribution.

$$\int_0^{\infty} \frac{\phi(x)+\phi(-x)-2\phi(0)}{x^2}\,dx\ ;\quad PV\!\int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{x^2}\,dx\ ;\quad \text{NOT}\ \cancel{2PV\!\int_0^{\infty} \frac{\phi(x)-\phi(0)}{x^2}\,dx}\ !$$

A direct calculation of the Fourier transform requires the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin(\omega x)}{x}\right)^2 dx = \pi |\omega|$$

It follows from

$$\widehat{\mathbb{1}_{[-1,1]}(x)} = \sqrt{\frac{2}{\pi}} \, \frac{\sin(\omega)}{\omega}$$

via Plancherel's identity,

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left|\frac{\sin(\omega)}{\omega}\right|^2 d\omega = \int_{-1}^{1} 1^2 dx = 2.$$

In other words $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx = \pi.$

Now replace x by ωx (and be careful!).

The FOURIER TRANSFORM is

$$\langle \widehat{x^{-2}}, \phi \rangle = \langle x^{-2}, \hat{\phi} \rangle$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\hat{\phi}(x) + \hat{\phi}(-x) - 2\hat{\phi}(0)}{x^2} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\omega x} + e^{+i\omega x} - 2}{x^2} \phi(\omega) \frac{d\omega}{\sqrt{2\pi}} dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\omega) \left(\int_{-\infty}^{\infty} \frac{\left(e^{i\frac{\omega x}{2}} - e^{-i\frac{\omega x}{2}}\right)^2}{x^2} dx \right) d\omega$$

$$= \frac{(2i)^2}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\omega) \underbrace{\left(\int_{-\infty}^{\infty} \frac{\left(\sin(\frac{\omega x}{2})\right)^2}{x^2} dx \right)}_{= \pi \left|\frac{\omega}{2}\right|} d\omega$$

$$= -\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} \phi(\omega)\, |\omega| \, d\omega$$

Thus $\widehat{x^{-2}} = -\sqrt{\frac{\pi}{2}}\, |\omega|$

$\widehat{|x|} = -\sqrt{\frac{2}{\pi}}\, \omega^{-2}$ (inverse)

The derivative of

$$\left\langle PV(\tfrac{1}{x}), \phi \right\rangle = \lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} \frac{\phi(x)}{x} dx$$

of course $\frac{d}{dx} x^{-1} = -x^{-2}$ when $x \neq 0$

is indeed $-x^{-2}$. To see this

$$\left\langle \frac{d}{dx} PV(\tfrac{1}{x}), \phi \right\rangle \overset{\text{def.}}{=} -\left\langle PV(\tfrac{1}{x}), \phi' \right\rangle$$

$$\overset{\text{def.}}{=} - \lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} \frac{\phi'(x)}{x} dx \qquad \phi'(x) = \frac{d}{dx}\left[\phi(x) - \phi(0)\right]$$

$$\int_{\varepsilon}^{\infty} \frac{\phi'(x)}{x} dx = \Big|_{\varepsilon}^{\infty} \frac{\phi(x) - \phi(0)}{x} + \int_{\varepsilon}^{\infty} \frac{\phi(x) - \phi(0)}{x^2}$$

$$\int_{|x| \geq \varepsilon} \frac{\phi'(x)}{x} dx = \underbrace{- \frac{\phi(\varepsilon) + \phi(-\varepsilon) - 2\phi(0)}{\varepsilon}}_{\to 0 \text{ as } \varepsilon \to 0} + \int_{|x| \geq \varepsilon} \frac{\phi(x) - \phi(0)}{x^2} dx$$

But

$$\int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{x^2}\,dx = \int_{-\infty}^{\infty} \frac{\phi(-x)-\phi(0)}{x^2}\,dx$$

Thus

$$+\lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\phi'(x)}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\phi(x)+\phi(-x)-2\phi(0)}{x^2}\,dx$$

$$= \langle x^{-2}, \phi \rangle .$$

This proves the differentiation formula.

If we know that

$$\widehat{PV\left(\tfrac{1}{x}\right)} = -i\sqrt{\tfrac{\pi}{2}}\,\mathrm{sign}(\omega) \qquad *)$$

we get

$$\widehat{-x^{-2}} = +i\omega\,\left(-i\sqrt{\tfrac{\pi}{2}}\right)\mathrm{sign}(\omega) \qquad \text{RULE FOR } \widehat{f'}.$$

$$= +\sqrt{\tfrac{\pi}{2}}\,|\omega|$$

and again

$$\widehat{x^{-2}} = -\sqrt{\tfrac{\pi}{2}}\,|\omega|$$

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*) Now a direct calculation requires

$$\int_0^{\infty} \frac{\sin(\omega x)}{x}\,dx = \frac{\pi}{2}\,\mathrm{sign}(\omega)$$