THE CENTRAL LIMIT THEOREM

Let $X_1, X_2, X_3, \ldots$ be identically distributed, independent random variables, say

$$\begin{cases} P(a < X_j \le b) = \int_a^b f(x)\,dx \\ E(X_j) = \int_{-\infty}^{\infty} x\, f(x)\,dx = m \quad \text{mean} \qquad j = 1, 2, 3, \ldots \\ V(X_j) = E(|X_j - m|^2) = \sigma^2 \quad \text{variance} \end{cases}$$

Then

$$P\left(a < \frac{\overbrace{X_1 + \cdots + X_n}^{S_n} - nm}{\sqrt{n}\,\sigma} \le b\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}}\,dx$$

Proof: We may assume $f \ge 0$,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x\, f(x)\,dx = 0, \quad \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1$$

$$m = 0 \qquad \qquad \sigma = 1$$

Denote

$$f^{(n)} = \overbrace{f * f * \cdots * f}^{n \text{ times}}. \qquad (\text{convolution!})$$

We observe that

$$P(a < \underbrace{X_1 + X_2 + \cdots + X_n}_{S_n} \le b) =$$

$$= \iint_{a < x_1 + x_2 + \cdots + x_n \le b} \cdots \int f(x_1) f(x_2) \cdots f(x_n) \, dx_1 \, dx_2 \cdots dx_n$$

Verify this first for $n = 2$.

$$= \int_a^b f^{(n)}(x) \, dx \qquad (1 \text{ variable!})$$

(To see this, recall that by the independency

$$P(a_1 < X_1 \le b_1 ; a_2 < X_2 \le b_2 ; \cdots ; a_n < X_n \le b_n)$$

$$= \int_{a_1}^{b_1} f(x_1) dx_1 \int_{a_2}^{b_2} f(x_2) dx_2 \cdots \int_{a_n}^{b_n} f(x_n) dx_n .)$$

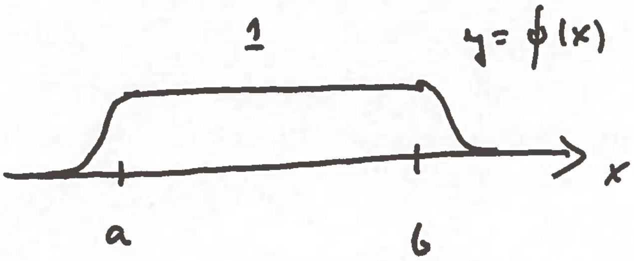
Thus

$$P\left(a < \frac{S_n}{\sqrt{n}} \le b\right) = \int_{a\sqrt{n}}^{b\sqrt{n}} f^{(n)}(x) \, dx$$

$$= \sqrt{n} \int_a^b f^{(n)}(\sqrt{n}\, x) \, dx \xrightarrow[\text{CLAIM}]{?} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx$$

as $n \to \infty$ Gauß

Choose a cut-off function $\phi(x)$ as



Has to approximate $\chi_{[a,b]} = \begin{cases} 1 & \text{in } [a,b] \\ 0 & \text{otherwise} \end{cases}$

$$\sqrt{n}\int_{-\infty}^{\infty} f^{(n)}(\sqrt{n}\,x)\,\phi(x)\,dx = \int_{-\infty}^{\infty} \sqrt{n}\, f^{(n)}(\sqrt{n}\,x)\, \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}(\omega) e^{i\omega x}\, d\omega}_{\phi(x)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}(\omega)\underbrace{\left(\int_{-\infty}^{\infty}\sqrt{n}\, f^{(n)}(\sqrt{n}\,x)e^{i\omega x}\,dx\right)}_{\int_{-\infty}^{\infty} f^{(n)}(y)\, e^{i\frac{\omega}{\sqrt{n}}y}\,dy} d\omega$$

$$= \int_{-\infty}^{\infty}\hat{\phi}(\omega)\,\overset{\text{conjugate}}{\overline{\widehat{f^{(n)}}\left(\tfrac{\omega}{\sqrt{n}}\right)}}\,d\omega$$

The conjugate fixes "the wrong sign of $+i$".

$$\widehat{f * f * \cdots * f} = \sqrt{2\pi}^{\,n-1}\,\hat{f}^{\,n}$$

$$= \sqrt{2\pi}^{\,n-1}\int_{-\infty}^{\infty}\hat{\phi}(\omega)\,\overline{\left(\hat{f}\left(\tfrac{\omega}{\sqrt{n}}\right)\right)^{n}}\,d\omega$$

$$= \sqrt{2\pi}^{\,n-1}\int_{-\infty}^{\infty}\hat{\phi}(\omega)\left(\overline{\hat{f}}\left(\tfrac{\omega}{\sqrt{n}}\right)\right)^{n}d\omega$$

$$\overline{\hat{f}}\left(\frac{\omega}{\sqrt{n}}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\frac{\omega}{\sqrt{n}}x}\, f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1 + i\frac{\omega x}{\sqrt{n}} - \frac{\omega^2 x^2}{2n}\left(1+\varepsilon_n(x,\omega)\right)\right) f(x)\,dx$$

$$\approx \frac{1}{\sqrt{2\pi}}\left(1 + \underset{m=0}{0} - \frac{\omega^2}{2n}\right)$$

$\sigma = 1$

Thus

$$\sqrt{n}\int_{-\infty}^{\infty} f^{(n)}(\sqrt{n}\,x)\,\phi(x)\,dx$$

$$\approx \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}(\omega)\left(1-\frac{\omega^2}{2n}\right)^n d\omega$$

$$\longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}}\,\hat{\phi}(\omega)\,d\omega \qquad \text{(Plancherel)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,\phi(x)\,dx$$

$$\approx \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{-x^2/2}\,dx$$

In toto

$$P\left(a < \frac{S_n}{\sqrt{n}} \le b\right) = \sqrt{n}\int_a^b f^{(n)}(\sqrt{n}\,x)\,dx$$

$$\approx \sqrt{n}\int_{-\infty}^{\infty} f^{(n)}(\sqrt{n}\,x)\,\phi(x)\,dx \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\phi(x)\,dx$$

$$\approx \frac{1}{\sqrt{2\pi}}\int_{-a}^{b} e^{-x^2/2}\,dx$$

Remarks: All the $\approx$ can be exactly justified.