COMPLETENESS OF THE TRIGONOMETRIC SYSTEM IN $L^2(-\pi,\pi)$

$$e_k = \frac{1}{\sqrt{2\pi}} e^{ikx} \qquad (k = 0, \pm 1, \pm 2, \dots)$$

$$\langle e_k, e_j \rangle = \int_{-\pi}^{\pi} e_k(x)\overline{e_j(x)}\, dx = \delta_{kj}$$

THEOREM If $f \in L^2(-\pi,\pi)$, then

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| f(x) - \sum_{n=-N}^{N} \langle f, e_n \rangle e_n(x) \right|^2 dx = 0$$

The Fourier series converges in L^2.

Proof: Assume that $f \in L^2(-\pi,\pi)$ and that

$$\langle f, e_n \rangle = \int_{-\pi}^{\pi} f(x) e^{-inx} \frac{dx}{\sqrt{2\pi}} = 0$$

when $n = 0, \pm 1, \pm 2, \dots$. We only have to prove that $f = 0$. To this end, we use a continuous function:

$$F(t) = \int_{-\pi}^{t} f(x)\, dx - C, \qquad -\pi \le t \le \pi.$$

Now $\langle F, e_n \rangle = 0$, when $n = \pm 1, \pm 2, \dots$ because

$$\sqrt{2\pi}\langle F, e_n \rangle = \int_{-\pi}^{\pi} e^{-int} \left\{ \int_{-\pi}^{t} f(x)dx - C \right\} dt =$$

$$= \int_{-\pi}^{\pi} dt \int_{-\pi}^{t} e^{-int} f(x)\, dx = \int_{-\pi}^{\pi} f(x) \left(\int_{x}^{\pi} e^{-int} dt \right) dx$$

$$= \int_{-\pi}^{\pi} \frac{e^{-in\pi} - e^{-inx}}{-in} f(x)\, dx = \frac{(-1)^n}{-in} \int_{-\pi}^{\pi} f(x) dx$$

$\langle f, e_0 \rangle = 0$

$$+ \frac{1}{in} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = 0 - 0 \quad (n \neq 0)$$

by assumption.

Choose c so that also $\langle F, e_0 \rangle = 0$, i.e.

$$\int_{-\pi}^{\pi} F(t) dt = \int_{-\pi}^{\pi} \left(\int_{-\pi}^{t} f(x) dx \right) dt - 2\pi c = 0.$$

Since $F(\pi) = F(-\pi) = -c$ (recall that $\int_{-\pi}^{\pi} f dx = 0$) there is, given $\varepsilon > 0$, a trigonometric polynomial $T_\varepsilon(x) = \sum_{k=-N_\varepsilon}^{N_\varepsilon} A_k^\varepsilon e^{ikx}$ such that

$$\max_{|x| \le \pi} |F(x) - T_\varepsilon(x)| \le \varepsilon$$

(Fejér's Theorem; take the Cesàro sums.) Now

$$(\dagger) \qquad \langle F, T_\varepsilon \rangle = 0. \qquad \text{(Recall } \langle F, e_n \rangle = 0 \text{ for all } n.)$$

$$\int_{-\pi}^{\pi} |F|^2 dx = \int_{-\pi}^{\pi} F \bar{F} dx \overset{(\dagger)}{=} \int_{-\pi}^{\pi} F(\bar{F} - \bar{T}_\varepsilon) dx \le$$

CAUCHY

$$\leq \left\{ \int_{-\pi}^{\pi} |F|^2 dx \right\}^{1/2} \left\{ \int_{-\pi}^{\pi} |F - T_\varepsilon|^2 dx \right\}^{1/2}$$

$$\leq \varepsilon \sqrt{2\pi} \left\{ \int_{-\pi}^{\pi} |f|^2 dx \right\}^{1/2}$$ and it follows that

$$\int_{-\pi}^{\pi} |F|^2 dx \leq 2\pi \varepsilon^2$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\int |F|^2 dx = 0$ and $F(x) \equiv 0$. Therefore

$$\int_{-\pi}^{t} f(x)\, dx = c, \text{ when } -\pi \leq t \leq \pi.$$

(In particular, $c = 0$.) This is possible only if $f = 0$ in L^2 ($f(x) = 0$ except possibly in a set of length measure zero.)

A FOOTNOTE. The set E of points in $\mathbb{R}$ is of <u>measure zero</u>, if, given $\varepsilon > 0$, there are intervals $I_j^\varepsilon = (a_j^\varepsilon, b_j^\varepsilon)$ such that

$$E \subset \bigcup_{j=1}^{\infty} I_j^\varepsilon \quad \text{and} \quad \sum_{j=1}^{\infty} (b_j^\varepsilon - a_j^\varepsilon) < \varepsilon.$$

(COVERING) SUM OF LENGTHS

At most a countable number of intervals!

<u>Notation</u>: $$\text{mes}(E) = \int_E dx = 0.$$

Each point $x \in E$ belongs to some interval I_j^ε.

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy, \text{ if } \int_a^b \int_c^d |f(x,y)|\, dy\, dx < \infty$$

ABS. CONV.