THE SHANNON WAVELET (§5.4 page 223)

$$\varphi(x) = \frac{\sin \pi x}{\pi x} \quad , \quad \hat{\varphi}(\omega) = \frac{1}{\sqrt{2\pi}} \mathbb{1}_{(-\pi,\pi)}(\omega)$$

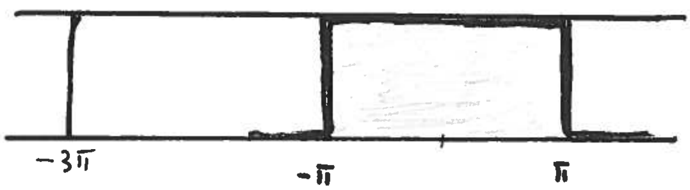
1. The system $\varphi(x-k)$, $k \in \mathbb{Z}$, is <u>orthonormal</u>, because

$$\langle \varphi(x-k), \varphi(x-\ell) \rangle = \langle e^{-i\omega k} \hat{\varphi}(\omega), e^{-i\omega \ell} \hat{\varphi}(\omega) \rangle$$

$$= \int e^{-i\omega k} e^{+i\omega \ell} |\hat{\varphi}(\omega)|^2 d\omega = \int_{-\pi}^{\pi} \frac{e^{i\omega(\ell-k)}}{2\pi} d\omega$$

$= \delta_{\ell k}$. (The orthonormality also follows from

$$2\pi \sum |\hat{\varphi}(\omega + 2k\pi)|^2 = \sum \mathbb{1}^2_{(-\pi,\pi)}(\omega + 2k\pi) = 1 \quad \text{a.e.})$$



Define

2. $$V_j = \{ f \in L^2(\mathbb{R}) \mid \hat{f}(\omega) = 0, \text{ when } |\omega| > 2^j \pi \}$$

It is clear that

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

If $f \in \bigcap V_j$, then $\hat{f}(\omega) = 0$, when $\omega \neq 0$ and hence $f = 0$. In other words

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\}.$$

The fact that $L^2(\mathbb{R}) = \overline{\bigcup_{j=-\infty}^{\infty} V_j}$

See 3 about V_0!

can be seen in the following way. Let $f \in L^2$ and define

$$f_j(x) = \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} \hat{f}(\omega) e^{+i\omega x} d\omega$$

Then the Fourier transform is

$$\hat{f}_j(\omega) = \begin{cases} \hat{f}(\omega), & \text{when } |\omega| \le 2^j\pi, \\ 0, & \text{when } |\omega| > 2^j\pi, \end{cases}$$

because $f_j(x)$ was defined as the inverse transform of this function. We have $f_j \in V_j$ and

$$\| f - f_j \|_2^2 = \| \hat{f} - \hat{f}_j \|_2^2 = \int_{|\omega| > 2^j\pi} |\hat{f}(\omega)|^2 d\omega$$

$$\longrightarrow 0 \text{ as } j \to +\infty,$$

because $\hat{f} \in L^2$. This proves the desired density.

(3) By Shannons sampling theorem any $f \in V_0$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)} = \sum_{-\infty}^{\infty} f(n)\, \phi(x-n)$$

Thus we have a basis of translated functions.

(4) Since $\varphi \in V_1$, we can again use Shannons theorem: $\phi\left(\frac{x}{2}\right) = \sum \phi\left(\frac{k}{2}\right) \frac{\sin \pi(x-k)}{\pi(x-k)}$. replace then x by $2x$

$$\phi(x) = \phi(2x) + \sum_{k \ne 0} \phi\left(\frac{k}{2}\right) \frac{\sin(2\pi x - k\pi)}{2\pi x - k\pi}$$

(The term $k=0$ separate.)

Here every second term is zero and we arrive at

Take $f(x) = \phi\left(\frac{x}{2}\right)$ $\quad \widehat{\phi\left(\frac{x}{2}\right)} = 2\hat{\phi}(2\omega) = 0$, when $|\omega| > \frac{\pi}{2}$

and therefore when $|\omega| > \pi$

(†) $$\phi(x) = \phi(2x) + \sum_{k=-\infty}^{\infty} \frac{2(-1)^k}{(2k+1)\pi}\, \phi(2x-2k-1)$$

⑤ Finally, we need an expression for

$$\psi(x) = \sum (-1)^{\ell}\, \overline{p_{1-\ell}}\, \phi(2x-\ell)$$

$$p_{1-\ell} = \begin{cases} 1, & \ell = 1 \\ 0, & \ell = \text{odd}. \end{cases}$$

$$1-\ell = 2k+1, \quad (-1)^{\ell} p_{1-\ell} = \frac{2(-1)^k}{\pi(2k+1)} \qquad (\ell \text{ even})$$

$$\psi(x) = -\phi(2x-1) + \sum \frac{2(-1)^k}{\pi(2k+1)}\, \phi(2x+2k)$$

$$= -\phi(2x-1) - \sum \frac{2(-1)^k}{\pi(2k-1)}\, \phi(2x-2k)$$

Remark: Replace k by $k+1$, compare with (†) to obtain the sum. Then you obtain

$$\boxed{\psi(x) = -2\, \frac{\cos(\pi x) - \sin(2\pi x)}{\pi(2x-1)}}$$

⑥ Scaling property.

$$\widehat{f(2^j x)} = 2^{-j}\, \hat{f}(2^{-j}\omega) \quad \text{or} \quad \widehat{2^{j/2} f(2^j \cdot)} = 2^{-j/2}\, \hat{f}(2^{-j}\omega)$$

$$f \in V_0 \iff \hat{f}(\omega) = 0 \text{ when } |\omega| > \pi$$

$$\iff \hat{f}(2^{-j}\omega) = 0 \text{ when } |\omega| > 2^j\pi$$

$$\iff f(2^j \cdot) \in V_j$$