The Example with Poisson's summation

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$$\begin{cases} f(x) = e^{-2\pi a|x|}, & a > 0 \\ \hat{f}(\xi) = \dfrac{a}{\pi(x^2+a^2)} & \text{(by direct integration)} \end{cases}$$

$$\boxed{\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} f(n+x) \qquad \text{Poisson's Summation Formula}}$$

In this case

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\frac{1}{a\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{a}{n^2+a^2}\cos(2n\pi x) = \sum_{n=-\infty}^{\infty}\frac{a}{\pi(n^2+a^2)}e^{2\pi i n x}$$

$$\overset{\text{POISSON}}{=} \sum_{n=-\infty}^{\infty} e^{-2\pi a|x+n|}$$

To evaluate the geometric series, let $0 \le x < 1$. Then

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|x+n|} = \sum_{n=0}^{\infty} e^{-2\pi a(x+n)} + \sum_{n=-\infty}^{-1} e^{2\pi a(x+n)} \qquad (n=-m)$$

$$= e^{-2\pi a x}\frac{1}{1-e^{-2\pi a}} + e^{2\pi a x}\sum_{m=1}^{\infty} e^{-2\pi a m}$$

$$= e^{-2\pi a x}\frac{1}{1-e^{-2\pi a}} + e^{2\pi a x}e^{-2\pi a}\frac{1}{1-e^{-2\pi a}}$$

$$= \frac{e^{\pi a}e^{-2\pi a x} + e^{-\pi a}e^{2\pi a x}}{e^{\pi a}-e^{-\pi a}} = \frac{\cosh[\pi a(1-2x)]}{\sinh(\pi a)}$$

Write $2\pi x = y$ to get the desired formula