THE 2nd MOMENT FOR DAUBECHIE'S WAVELET

$$\phi(x) = \sum_{k=0}^{3} p_k \phi(2x-k) \qquad \begin{cases} p_0 = \frac{1+\sqrt{3}}{4} \\ p_1 = \frac{3+\sqrt{3}}{4} \\ p_2 = \frac{3-\sqrt{3}}{4} \\ p_3 = \frac{1-\sqrt{3}}{4} \end{cases}$$

$$\psi(x) = \sum (-1)^k \overline{p_{1-k}} \phi(2x-k)$$

$$= -p_0 \phi(2x-1) + p_1 \phi(2x) - p_2 \phi(2x+1) + p_3 \phi(2x+2)$$

$$= \sum_{k=-2}^{1} q_k \phi(2x-k)$$

$$\hat{\psi}(\omega) = \hat{\phi}\left(\frac{\omega}{2}\right) \underbrace{\left\{ \frac{1}{2} p_3 e^{\frac{2i\omega}{2}} - p_2 \cdot \frac{1}{2} e^{i\frac{\omega}{2}} + p_1 - \frac{1}{2} p_0 e^{-i\frac{\omega}{2}} \right\}}_{Q\left(\frac{\omega}{2}\right)}$$

By direct calculation $\quad Q(0) = 0, \quad Q'(0) = 0$

and

$$\frac{d^2}{d\omega^2} Q\left(\frac{\omega}{2}\right) = \frac{1}{8}\left\{ p_0 e^{-i\frac{\omega}{2}} + p_2 e^{i\frac{\omega}{2}} + 4 p_3 e^{i\frac{2\omega}{2}} \right\}$$

$$= \frac{\sqrt{3}}{8} \quad \text{when } \omega = 0$$

Thus

$$\hat{\psi}(0) = 0, \qquad \hat{\psi}'(0) = 0,$$

$$\hat{\psi}''(0) = \cdots = \frac{1}{8}\sqrt{3}\, \hat{\phi}(0)$$

$$= \frac{\sqrt{3}}{8} \frac{1}{\sqrt{2\pi}} \int \phi(x)\,dx = \frac{\sqrt{3}}{8 \cdot \sqrt{2\pi}}.$$

To calculate the moments, note that

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-i\omega x} dx$$

$$\hat{\psi}'(\omega) = \frac{-i}{\sqrt{2\pi}} \int x\, \psi(x) e^{-i\omega x} dx$$

$$\hat{\psi}''(\omega) = \frac{(-i)^2}{\sqrt{2\pi}} \int x^2 \psi(x)\, e^{-i\omega x} dx$$

and so, set $\omega = 0$,

$$\int_{-\infty}^{\infty} \psi(x)\, dx = \underline{\underline{0}}, \quad \int_{-\infty}^{\infty} x\, \psi(x)\, dx = \underline{\underline{0}}$$

$$-\frac{1}{\sqrt{2\pi}} \int x^2 \psi(x)\, dx = \hat{\psi}''(0) = \frac{\sqrt{3}}{8\sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty} x^2 \psi(x)\, dx = \underline{\underline{-\frac{\sqrt{3}}{8}}}\, .$$

<u>Remark</u> All higher moments can be calculated in this way.

The counterexample in Stein 6, §7, Problem 7c comes from

$$x^2 (-\Delta)^{\frac{1}{2}} f(x) = \int 2\pi |\xi| \hat{f}(\xi) e^{+2\pi i \xi x} d\xi \cdot x^2$$

$$= -\frac{1}{2\pi} \int |\xi| \hat{f}(\xi) \frac{d^2}{d\xi^2} \left(e^{2\pi i x \xi}\right) d\xi \qquad \Delta = \frac{d^2}{dx^2}$$