FOURIER ANALYSIS TMA4170

8. VIII. 2013

(1) $$\cosh(x) = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{1+n^2}e^{inx} \qquad (|x|\le\pi)$$

By Parseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh^2(x)\,dx = \frac{\sinh^2(\pi)}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(1+n^2)^2}$$

$$= \frac{\sinh^2(\pi)}{\pi^2}\left\{1+2\sum_{n=1}^{\infty}\frac{1}{(1+n^2)^2}\right\}$$

Now

$$\int_{-\pi}^{\pi}\cosh^2(x)\,dx = \frac{1}{4}\int_{-\pi}^{\pi}\left(e^{2x}+2+e^{-2x}\right)dx$$

$$= \cdots = \pi + 2\sinh(2\pi)$$

It follows that the desired sum is

$$\sum_{n=1}^{\infty}\frac{1}{(1+n^2)^2} = -\frac{1}{2}+\frac{\pi^2}{4\sinh^2(\pi)}+\frac{\pi\sinh(2\pi)}{2\sinh^2(\pi)}$$

(2) We use

$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$$

Let $f(x) = e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}$. Then $(a = 1/2)$

$$\widehat{(f * f * \cdots * f)(x)} = \sqrt{2\pi}^{\,n-1} \frac{1}{\sqrt{2\pi}^{\,n}} \left[e^{-\frac{\omega^2}{2}} \right]^n$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{n\omega^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi n}} \cdot \underbrace{\frac{1}{\sqrt{1/n}} e^{-\frac{\omega^2}{4(1/2n)}}}_{\text{"}a = 1/2n\text{"}} .$$

Taking the inverse transform we obtain the answer

$$(f * f * \cdots * f)(x) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{2n}} .$$

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(3) This is Fejer's kernel

$$\frac{1 + (e^{-ix} + 1 + e^{ix}) + \cdots + \sum_{-N}^{N} e^{inx}}{N+1}$$

$$= \frac{1}{N+1} \left(\frac{\sin\left(\frac{N+1}{2} x\right)}{\sin\left(\frac{x}{2}\right)} \right)^2 \geq 0 \quad \text{(a square!)}$$

There are many proofs of the formula.

(4) In general, the formula

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle , \qquad \phi \in \mathcal{S}(\mathbb{R})$$

defines $\hat{T}$. In this case

$$\langle \hat{T}, \phi \rangle = \int_{-\infty}^{\infty} e^{2\pi i x} \hat{\phi}(x)\, dx = \phi(1)$$

$$= \langle \delta_1 , \phi \rangle$$

Hence $\hat{T}(\phi) = \delta_1(\phi)$; Dirac's delta. One writes

$$\widehat{e^{2\pi i x}} = \delta_1(\xi).$$

(5) Since $\psi(x)$ is of compact support, $\psi = 0$ outside some interval $[a, b]$. Given $\varepsilon > 0$ there is a <u>polynomial</u> $P_\varepsilon(x)$ such that $\max_{a \le x \le b} |\psi(x) - P_\varepsilon(x)| < \varepsilon$ (Weierstrass' approximation theorem). The assumption on the moments implies that

$$\int_a^b \psi(x) P_\varepsilon(x)\, dx = 0.$$

Now

$$\int_{-\infty}^{\infty} \psi(x)^2 dx = \int_a^b (\psi(x) - P_\varepsilon(x))^2 dx - \int_a^b P_\varepsilon(x)^2 dx$$

$$\leq \int_a^b (\psi(x) - P_\varepsilon(x))^2 \, dx \leq \varepsilon^2 (b-a)$$

Since $\varepsilon > 0$ was arbitrary, $\int_{-\infty}^{\infty} \psi(x)^2 dx = 0$.

Thus $\psi(x) \equiv 0$. ∎

(6) We can expand $\hat{f}(\omega)$ in a Fourier series of period 2π, say

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{in\omega} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{in\omega} d\omega = \frac{1}{\sqrt{2\pi}} f(+n)$$

By Parseval's formula and Plancherel's identity

$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

The desired result follows.