FOURIER ANALYSIS 8/2013

① Verify that

$$\lim_{\varepsilon \to 0+} \int_{|x| \geq \varepsilon} \frac{\phi(x)\,dx}{x} = -\int_{-\infty}^{\infty} \phi'(x) \ln(|x|)\,dx$$

for all test functions $\phi \in C_0^\infty(\mathbb{R})$.

② Verify that

$$\Delta\left(\frac{1}{r}\right) = -c\,\delta \qquad \left(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\right)$$

in the sense of distributions. Determine the constant c. (δ = Dirac's δ.)

③ Find a fundamental solution for $L = \frac{d}{dx} - a$ on $\mathbb{R}$, i.e., solve

$$\frac{du}{dx} - au = \delta.$$

Then, show that

$$v(x) = \begin{cases} a^{-1}\sinh(ax), & x > 0 \\ 0, & x < 0 \end{cases}$$

is a fundamental solution for $\frac{d^2}{dx^2} - a^2$.

④[+)] Show that in $\mathbb{R}^3$ the function

$$F(x) = -\frac{1}{4\pi|x|}e^{-|x|}$$

(the YUKAWA potential) is the fundamental solution of $\Delta - I$, i.e.,

$$\Delta F - F = \delta$$

<u>Hint:</u> • The inverse Fourier transform of $-(1+4\pi^2|\xi|^2)^{-1}$.

• $$\iint_{|\xi|=1} e^{2\pi i\langle\xi,x\rangle}\,dS(\xi) = \frac{2\sin(2\pi|x|)}{|x|}.$$

• $$\widehat{\frac{t}{\pi(t^2+a^2)}} = e^{-2\pi a|\tau|}\,\frac{\operatorname{sign}(\tau)}{i} \qquad (a>0)$$

⑤ $$PV\left(\tfrac{1}{x}\right)(\varphi) \overset{\text{DEF.}}{=} \lim_{\varepsilon\to 0+}\int_{|x|\geq\varepsilon}\varphi(x)\,\frac{dx}{x} \qquad \text{(one dimension)}$$

Show that this is the same as $\displaystyle\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{x\,\varphi(x)}{x^2+\varepsilon^2}\,dx$.

Then find the Fourier transform of this distribution.

<u>Answer:</u> $\widehat{PV\left(\tfrac{1}{x}\right)} = \frac{\pi}{i}\operatorname{sign}(\xi)$