FOURIER ANALYSIS
Øv. 4 (18.II.2013)

① Verify the expansions

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2} \qquad (0 < x < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} \sin(nx) = \begin{cases} \frac{1}{2}(\pi - 1)x, & 0 \le x \le 1, \\ \frac{1}{2}(\pi - x), & 1 \le x \le \pi. \end{cases}$$

Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \sum_{n=1}^{\infty} \left(\frac{\sin(n)}{n}\right)^2.$$

② If γ is irrational,

$$\frac{f(\gamma) + f(2\gamma) + \cdots + f(N\gamma)}{N} \longrightarrow \int_0^1 f(x)\,dx$$

for any $f \in C(\mathbb{R})$ or C period 1. (Weyl's equidistribution theorem.) Let now $\gamma = \frac{355}{113}$. Exhibit a function f such that the limit is not the integral.

FRACTIONAL PART

③ Show that $\langle \log(n) \rangle$, $n = 1, 2, 3, \ldots$, is not equidistributed in $[0,1)$. <u>Hint</u> A simple instance of the Euler–Maclaurin summation formula can be helpful to estimate

$$\sum_{n=1}^{N} e^{2\pi i k \log(n)} - \int_1^N e^{2\pi i k \log(x)}\,dx,$$

$k = 1$.

(4) Chapter 5, Exercise 2.

(5) Show by examples that

$$L^1(\mathbb{R}) \not\subset L^2(\mathbb{R}),$$

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<u>Remark</u>: $f \in L^p(\Omega) \iff$

$$\|f\|_{p,\Omega} = \left\{ \int_\Omega |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty$$

Usually, $1 \leq p < \infty$.

(6) Suppose that $\varphi : [a,b] \longrightarrow \mathbb{R}$ is continuous. Verify that

$$\lim_{p \to \infty} \left\{ \int_a^b |\varphi(x)|^p dx \right\}^{\frac{1}{p}} = \max_{a \leq x \leq b} |\varphi(x)| .$$

<u>Remark</u>: In general, the limit is

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_{a<x<b} |\varphi(x)| .$$