FOURIER ANALYSIS
Øv. 10

① Suppose that $\psi \in C_0^\infty(\mathbb{R})$ and that all its moments vanish:

$$\int_{-\infty}^{\infty} x^n \psi(x)\,dx = 0 \quad (n = 0, 1, 2, 3, \ldots)$$

Prove that $\psi = 0$. (Remark: The conclusion is not always valid under the weaker assumption $\psi \in C^\infty(\mathbb{R})$. The bounded support is needed.)

② Assume that the trigonometric polynomial

$$P(\omega) = \frac{1}{2} \sum p_k e^{-i\omega k}$$

satisfies

$$\begin{cases} |P(\omega)|^2 + |P(\omega+\pi)|^2 = 1 \\ P(0) = 1, \quad P(\pi) = 0. \end{cases}$$

Consider the iteration

$$\phi_n(x) = \sum_k p_k \phi_{n-1}(2x - k)$$

starting with $\phi_0 = \phi_{HAAR}$ (then

$$\hat{\phi}_0(\omega) = \frac{1 - e^{-i\omega}}{\sqrt{2\pi}\, i\omega} \quad)$$

Prove by induction that the orthonormality is preserved for the translates $\phi_n(x-j)$, i.e.,

$$\langle \phi_n(x-j), \phi_n(x-\ell) \rangle = \delta_{\ell j}.$$

(Remark: The $\hat{\phi}_n$'s converge uniformly to some function $\hat{\phi}$. Under the "extra assumption" that $P(\omega) \neq 0$ when $|\omega| \leq \frac{\pi}{2}$, one can also show that $\hat{\phi}_n \to \hat{\phi}$ in L^2. Then $\phi_n \to \phi$ in L^2 and one can pass to the limit in the formulas:

$$\phi(x) = \sum p_k \phi(2x-k), \quad \langle \phi_{0j}, \phi_{0\ell} \rangle = \delta_{j\ell}.)$$

③ Let $a_n \geq 0$. Prove that

$$\prod (1+a_n) \text{ converges} \iff \sum a_n \text{ converges}$$

④ Show that: $\sum |a_n| < \infty \implies \prod (1+a_n)$ converges.

⑤ $(1+q)(1+q^2)(1+q^4)\cdots = \,?$ $\quad (|q| < 1)$