FOURIERANALYSE

TMA 4170 — Peter Lindqvist

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① $$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$$

by Bessel's ineq. Thus the series converges and $|c_n|^2 \longrightarrow 0$. Then also $c_n \longrightarrow 0$.
(This is a special case of the Riemann–Lebesgue lemma.)

② $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} e^{-\frac{x^2}{2}}\,dx = e^{-\frac{\omega^2}{2}} \quad (\text{known})$$

Differentiate twice

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-ix)e^{-i\omega x} e^{-\frac{x^2}{2}}\,dx = -\omega e^{-\frac{\omega^2}{2}}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-x^2)e^{-i\omega x} e^{-\frac{x^2}{2}}\,dx = +\omega^2 e^{-\frac{\omega^2}{2}} - e^{-\frac{\omega^2}{2}}$$

It follows that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{(1-x^2)e^{-\frac{x^2}{2}}}_{g(x)} e^{-i\omega x}\,dx = \omega^2 e^{-\frac{\omega^2}{2}}$$

④ Stein, p. 190

$$\frac{1}{4\pi}\iint_{|\vec{\gamma}|=1} e^{2\pi i\langle x,\vec{\gamma}\rangle}\,dS(\vec{\gamma}) = \frac{\sin(2\pi|x|)}{2\pi|x|}.$$

Remark $\langle Ox, \gamma\rangle = \langle x, O^T\gamma\rangle$ $\quad O =$ orthonorm.

$\varkappa = O^T\gamma$, $\quad dS(\varkappa) = +1\, dS(\gamma)$

(3) $$\frac{d^2 u}{dx^2} - 4u = \delta \quad (\text{Dirac's } \delta)$$

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x)\, dx$$

$$(-\omega^2 - 4)\,\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}}$$

$$\hat{u}(\omega) = -\frac{1}{\sqrt{2\pi}} \frac{1}{4+\omega^2}$$

Using $\mathcal{F}(e^{-2|x|}) = \frac{1}{\sqrt{2\pi}} \frac{4}{4+\omega^2}$

we get

$$u(x) = -\frac{1}{4} e^{-2|x|}$$

<u>Remarks</u>: The general solution is

$$Ae^{2x} + Be^{-2x} - \frac{1}{4} e^{-2|x|},$$

from which one can extract, for example

$$u(x) = \frac{1}{2} H(x) \sinh(2x).$$

One can also use $H' = \delta$ and find the solution through some simple manipulations based on

$$\frac{d^2}{dx^2} - 4 = \left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} + 2\right)$$

⑤ $|\psi(x)| \le 50 e^{-|x|}$

$$\int_{-\infty}^{\infty} \psi(x)\,dx = 0 \quad (= \hat{\psi}(0))$$

$$\hat{\psi}(\omega) = \hat{\psi}(\omega) - \hat{\psi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{-i\omega x} - 1)\psi(x)\,dx$$

$|e^{i\theta} - 1| \le |\theta|.$

$$|\hat{\psi}(\omega)| \le \frac{1}{\sqrt{2\pi}} \int |e^{-i\omega x} - 1|\,|\psi(x)|\,dx$$

$$\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\omega x| \cdot 50 e^{-|x|}\,dx = \frac{100|\omega|}{\sqrt{2\pi}}$$

Split the integral:

$$\int_{|\omega| \le 1} \frac{|\hat{\psi}(\omega)|^2\,d\omega}{|\omega|} \le \frac{100^2}{2\pi} \int_{-1}^{1} |\omega|\,d\omega = \frac{100^2}{2\pi}$$

$$\int_{|\omega| \ge 1} \frac{|\hat{\psi}(\omega)|^2\,d\omega}{|\omega|} \le \int_{|\omega| \ge 1} |\hat{\psi}(\omega)|^2\,d\omega \le \int_{-\infty}^{\infty} |\hat{\psi}(\omega)|^2\,d\omega$$

$$\overset{\text{PLANCHEREL}}{=} \int_{-\infty}^{\infty} |\psi(x)|^2\,dx \le 50^2 \int_{-\infty}^{\infty} e^{-2|x|}\,dx = 50^2.$$

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2\,dx}{|\omega|} \le \frac{100^2}{2\pi} + 50^2 < \infty \quad \text{q.e.d..}$$

<u>Remark</u>: Appears in the wavelet transform as "C_ψ."

⑥ $\hat{\phi}(x) = \widehat{\frac{\sin(\pi x)}{\pi x}} = \frac{1}{\sqrt{2\pi}} \chi_{(-\pi,\pi)}(\omega)$

(The calculation is easy, if one starts with

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{+i\omega x} \, d\omega .)$$

$$\phi_k(x) = \phi(x-k), \quad \hat{\phi}_k(\omega) = e^{-i\omega k} \hat{\phi}(\omega)$$

$$\int_{-\infty}^{\infty} \phi_k(x) \overline{\phi_j(x)} \, dx = \int_{-\infty}^{\infty} \hat{\phi}_k(\omega) \overline{\hat{\phi}_j(\omega)} \, d\omega$$

$$= \int_{-\pi}^{\pi} e^{-i\omega k} e^{i\omega j} |\phi(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(j-k)} d\omega = \delta_{jk}$$

This is the desired orthogonality:

$$\langle \phi_k, \phi_j \rangle = \delta_{kj} .$$

<u>Remark:</u> One can also use

$$\sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = \cdots .$$