Faglig kontakt: .... (.........)

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FOURIERANALYSE TMA 4170 Sensurfrist VIII

Hjelpemidler: Bestemt kalkulator. Ett A4-ark stemplet fra Instituttet med valgfri paaskrift av studenten.

1. It is known that

$$\cosh(x) = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{1+n^2}e^{inx},$$

   when $-\pi \leq x \leq \pi$. Evaluate the sum

$$\sum_{n=1}^{\infty}\frac{1}{(1+n^2)^2}.$$

2. We define the convolution

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

   Find

$$f \star f \star \cdots \star f \qquad \text{for} \qquad f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

   explicitly (there are n factors in the convolution). Hint: Fourier transform.

3. Prove that

$$\frac{1 + (e^{-ix} + 1 + e^{ix}) + \cdots + \sum_{-N}^{N} e^{inx}}{N+1} \;\geq\; 0$$

   for all real x and $N = 1, 2, \cdots$.

4. First, define the Fourier transform $\widehat{T}$ of the distribution

$$T(\phi) = \int_{-\infty}^{\infty} e^{2\pi i x} \phi(x)\, dx,$$

$\phi \in \mathcal{S}(\mathbb{R})$. Then, compute $\widehat{T}$, i.e. $\widehat{e^{2\pi i x}}$.

5. Assume that $\psi \in C_0^\infty(\mathbb{R})$ has vanishing moments:

$$\int_{-\infty}^{\infty} x^n \psi(x)\, dx \quad = \quad 0, \qquad n = 0, 1, 2, 3, \cdots .$$

Prove that $\psi(x) \equiv 0$.

6. Assume that the functon f is in the Schwartz class $\mathcal{S}(\mathbb{R})$ and that it is "band-limited":

$$\hat{f}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx \quad = \quad 0 \qquad \text{when} \qquad |\omega| > \pi.$$

Establish the formula

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \quad = \quad \sum_{-\infty}^{\infty} |f(n)|^2.$$