THE FOURIER TRANSFORM

$$\mathrm{sign}(x) \longleftrightarrow -\frac{i}{\pi}\,\mathrm{PV}\left(\frac{1}{\xi}\right)$$

1°) $$\widehat{e^{-2\pi\varepsilon|x|}\,\mathrm{sign}(x)} = -\frac{i}{\pi}\,\frac{\xi}{\xi^2+\varepsilon^2}$$

A FUNCTION IN $L^1(\mathbb{R})$ — A FUNCTION IN $L^1(\mathbb{R})$

Indeed $$\int_{-\infty}^{\infty} e^{-2\pi\varepsilon|x|}\,\mathrm{sign}(x)\, e^{-2i\pi x\xi}\, d\xi$$

$$= \int_0^{\infty} e^{-2\pi x(\varepsilon+i\xi)}\,dx - \int_{-\infty}^{\infty} e^{2\pi x(\varepsilon-i\xi)}\,dx = \overset{\text{DO IT!}}{\cdots}$$

<u>Remark</u>: The regularization avoids difficulties in Fubini's Theorem, which occurs in $\langle \mathrm{sign}|x|, \hat{\varphi}\rangle$.

2°) $$e^{-2\pi\varepsilon|x|}\,\mathrm{sign}(x) \xrightarrow[\varepsilon\to0+]{\mathcal{S}'} \boxed{\mathrm{sign}(x)}$$

$$-\frac{i}{\pi}\,\frac{\xi}{\xi^2+\varepsilon^2} \xrightarrow[\varepsilon\to0+]{\mathcal{S}'} \boxed{-\frac{i}{\pi}\,\mathrm{PV}\left(\frac{1}{\xi}\right)}$$

For example,

$$\int_{-\infty}^{\infty} \frac{\xi\,\varphi(\xi)}{\xi^2+\varepsilon^2}\,d\xi = \int_0^{\infty} \frac{\xi}{\xi^2+\varepsilon^2}\,\overbrace{[\varphi(\xi)-\varphi(-\xi)]}^{\approx 2\varphi'(0)\xi}\,d\xi \xrightarrow{\varepsilon\to0} \int_0^{\infty} \frac{\varphi(\xi)-\varphi(-\xi)}{\xi}\,d\xi$$

3°) $T_j \to T$ & $\hat{T}_j \to S$ in $\mathcal{S}'$

$$\Longrightarrow \hat{T} = S \quad (T(\hat{\varphi}) = S(\varphi))$$

The result follows.