Peter Lindqvist

FOURIERANALYSE

TMA 4170 Exam. 8. VI. 2012

(1)

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi x} f(x)\,dx$$

$$= \int_{-1}^{1} e^{-2\pi i \xi x}\,dx = \cdots = \frac{\sin(2\pi\xi)}{\pi\xi}$$

$$\widehat{f*f*f}(\xi) = \hat{f}(\xi)^3 = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^3$$

$$= 8\left[\operatorname{sinc}(2\pi\xi)\right]^3.$$

(2) $$|c_n|^2 \geq \frac{1}{|n|} \quad (n = \pm 1, \pm 2, \dots) \implies$$

$$\sum_{-\infty}^{\infty} |c_n|^2 = +\infty$$ Thus there is no such $f \in L^2(-\pi,\pi)$ by the Riesz-Fischer theorem. (Alternatively, Bessel's inequality yields

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx \geq +\infty,$$

if f is measurable at all.)

(3) Assumption: $\langle T, \varphi' \rangle = 0$ for each $\varphi \in C_0^\infty(\Omega)$

Claim: $\langle T, \varphi \rangle = C_T \int_{-\infty}^{\infty} \varphi(x)\,dx$

Proof: Choose $\eta \in C_0^\infty(\Omega)$ such that $\int_{-\infty}^{\infty} \eta(x)\,dx = 1$. Denote $\langle T, \eta \rangle = C_T$.

The function
$$\psi(x) = \int_{-\infty}^{x} \varphi(y)\,dy - \int_{-\infty}^{x} \eta(y)\,dy \cdot \int_{-\infty}^{\infty} \varphi(y)\,dy$$
is in the class $C_0^\infty(\mathbb{R})$. Hence

$$0 = \langle T, \psi' \rangle = \left\langle T, \varphi - \eta \cdot \int_{-\infty}^{\infty} \varphi(y)\,dy \right\rangle$$
$$= \langle T, \varphi \rangle - \int_{-\infty}^{\infty} \varphi(y)\,dy \cdot \langle T, \eta \rangle \implies$$
$$\langle T, \varphi \rangle = C_T \int_{-\infty}^{\infty} \varphi(y)\,dy \qquad \square$$

(4) $\hat{g}_k(\omega) = e^{-i\omega k}\hat{g}(\omega)$

PLANCHEREL

$$\langle g_k, g_l \rangle = \langle \hat{g}_k, \hat{g}_l \rangle$$

$$= \int_{-\infty}^{\infty} e^{-i\omega k}\hat{g}(\omega)\, e^{+i\omega l}\,\overline{\hat{g}(\omega)}\, d\omega$$

$$= \int_{-\infty}^{\infty} e^{i\omega(l-k)} |\hat{g}(\omega)|^2 d\omega$$

$$= \sum_{n=-\infty}^{\infty} \int_{2n\pi}^{2n\pi+2\pi} e^{i\omega(l-k)} |\hat{g}(\omega)|^2 d\omega$$

$$= \sum \int_0^{2\pi} e^{i\omega(l-k)} |\hat{g}(\omega+2n\pi)|^2 d\omega$$

$$= \int_0^{2\pi} e^{i\omega(l-k)} \underbrace{\sum_{-\infty}^{\infty} |\hat{g}(\omega+2n\pi)|^2}_{=1/2\pi \text{ by assumption.}} d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega(l-k)} d\omega = \delta_{kl} \qquad \square$$

(5) Start the iteration

$$\phi_{n+1}(x) = \sum_{k=0}^{5} p_k \phi_n(2x-k)$$

with $\phi_0 = \phi_{HAAR} = \chi_{[0,1]}$.

$$\operatorname{supp} \phi_n = [a_n, b_n].$$

Then $\phi_{n+1}(x) = 0$ if $2x - k > b_n$ or $2x - k < a_n$ for $k = 0, 1, \ldots, 5$. Thus $b_{n+1} \leq \frac{b_n + 5}{2}$ and $a_{n+1} \geq \frac{a_n + 0}{2}$. As $n \longrightarrow \infty$ we get

$$\operatorname{supp} \phi \subset [0, 5].$$

Now

$$\psi(x) = \sum_{k=-4}^{1} \overline{p_{1-k}} (-1)^k \phi(2x-k).$$

(In a similar manner) we conclude that

$$\operatorname{supp} \psi = [-2, 3]$$

(or a subset of $[-2, 3]$).

⑥ <u>Def.</u>: $\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle$

for each $\varphi \in \mathcal{S}(\mathbb{R})$. Now

$$\langle T, \varphi \rangle = \int_{-\infty}^{\infty} e^{2ia\pi x} \varphi(x)\, dx$$

<u>Answer</u>: $\hat{T} = \delta_a$ (Dirac's delta centered at a.)

⑦

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = f(x) - c_0 = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{inx}$$

$$\int_{-\pi}^{\pi} \left| f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \right|^2 dx = \underline{\underline{2\pi \sum_{n \neq 0} |c_n|^2}}$$

$$f'(x) = \sum_{-\infty}^{\infty} c_n i n e^{inx}$$

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx = \underline{\underline{2\pi \sum_{n=-\infty}^{\infty} n^2 |c_n|^2}}$$

The inequality $\sum_{n \neq 0} |c_n|^2 \leq \sum n^2 |c_n|^2$ is obvious!

Equality $\Leftrightarrow$ $c_n = 0$ when $n = \pm 2, \pm 3, \ldots$. In other words:

$$f(x) = a + b \sin(x) + c \cos(x)$$