FOURIERANALYSE

Exercise 3 / 2012

(1) Expand $f(t) = e^{ixt}$ (x not an integer) in a Fourier series with period 2π. Deduce the formulas

$$\frac{\pi}{\sin(\pi x)} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2}$$

$$\pi \operatorname{ctg}(\pi x) = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2}$$

$$\frac{\pi^2}{(\sin(\pi x))^2} = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^2}$$

<u>Remark</u>: One can further obtain

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \qquad \text{(Euler)}$$

(2)
$$\sum_{k=1}^{n} \cos[(2k-1)t] = \frac{\sin(2nt)}{2\sin(t)}$$

(3) Let $f(t) = \cosh(at)$, $a > 0$, when $-\pi \leq t < \pi$ and extend it with period 2π. Compute the Fourier series (in terms of cosines). Show that the Fourier series converges uniformly to f in $\mathbb{R}$.

(4) Let

$$f(t) = \begin{cases} 1 - t^2, & -1 \leq t \leq 1, \\ 0, & |t| > 1 \end{cases}$$

and

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt.$$

Find $\hat{f}(\xi)$. Answer: $\frac{1}{\pi^2 \xi^2}\left(\frac{\sin(2\pi\xi)}{2\pi\xi} - \cos(2\pi\xi)\right)$.

(5) $f(x) = \frac{1}{a^2 + x^2}$, $a > 0$

$\hat{f}(\xi) = ?$ Hint: Residues