FOURIERANALYSE TMA4170

Exercise 2/2012

① Derive the formula

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi}{2} - \frac{x}{2} \qquad (0 < x < 2\pi)$$

and use it to find the sum

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} e^{2in\frac{x}{a}} \qquad (0 < x < a)$$

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② Expand $f(t) = t^2$ over the interval $[0,1]$ in terms of sines. What is the sum of the series? $t = 1$?

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③ Assume that $f_n \longrightarrow f$ in $L^p(-\pi, \pi)$, i.e.,

$$\lim_{n \to \infty} \left\{ \int_{-\pi}^{\pi} |f(t) - f_n(t)|^p dt \right\}^{1/p} = 0,$$

where $p \geq 1$ is some exponent. Prove that

$$\lim_{n \to \infty} c_k(f_n) = c_k(f) \qquad k = 0, \pm 1, \pm 2, \dots$$

for the Fourier coefficients.

(4) We know that the trigonometric system is a basis in $L^2(-\pi, \pi)$. Show that the system

$$\frac{\sin(nx)}{\sqrt{\frac{\pi}{2}}} \qquad (n = 1, 2, 3, \ldots)$$

is a basis in the Hilbert space $L^2(0, \pi)$.

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(5) The "functions" $\varphi_1, \varphi_2, \varphi_3, \ldots$ are orthogonal in the Hilbert space $\mathcal{H}$, i.e.,

$$\langle \varphi_j, \varphi_k \rangle = 0 \quad \text{when } j \neq k.$$

Show that $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is linearly independent.

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(6) Use the Legendre polynomials $P_0 = 1$, $P_1(t) = t$, $P_2(t) = (3t^2 - 1)/2$ to find the best approximation $c_0 + c_1 P_1(t) + c_2 P_2(t)$ of $f(t) = |t|$, $-1 \le t \le 1$, in the usual $L^2(-1, 1)$ - norm. (They are not normalized.)

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(7) Study the convergence of the sequence

$$f_n(x) = nx(1-x)^n, \quad n = 1, 2, 3, \ldots$$

$0 \le x \le 1$. Is the convergence <u>uniform</u> in $[0, 1]$?