THE FAST FOURIER TRANSFORM

An algorithm for calculating the discrete Fourier transform

$$\hat{X}(k) = \sum_{n=0}^{N-1} X(n) e^{-nk\frac{2\pi i}{N}}$$

The factor $\frac{1}{N}$ has been skipped!

$$k = 0, 1, 2, \ldots, N-1$$

using only about $N \log_2 N$ complex multiplications in place of N^2 such operations.

$$N = 4, \quad W = e^{-\frac{2\pi i}{4}}$$

$$\begin{cases} \hat{X}(0) = X(0)W^0 + X(1)W^0 + X(2)W^0 + X(3)W \\ \hat{X}(1) = X(0)W^0 + X(1)W^1 + X(2)W^2 + X(3)W^3 \\ \hat{X}(2) = X(0)W^0 + X(1)W^2 + X(2)W^4 + X(3)W^6 \\ \hat{X}(3) = X(0)W^0 + X(1)W^3 + X(2)W^6 + X(3)W^9 \end{cases}$$

In matrix notation

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

$$W = e^{-i\frac{2\pi}{4}} \qquad W^0 = 1,\ W^1,\ W^2,\ W^3,$$
$$W^4 = 1,\ W^5 = W^1,\ W^6 = W^2 \text{ etc}$$

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(1) \\ \hat{X}(2) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W^1 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

A direct calculation here requires

6 $4^2 = 16$ multiplications "N^2"

$4 \cdot 3 = 12$ additions "$N(N-1)$"

However, we can do better! The matrix can be factorized:

$$\begin{pmatrix} \hat{X}(0) \\ \hat{X}(2) \\ \hat{X}(1) \\ \hat{X}(3) \end{pmatrix} = \begin{pmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}$$

"scrambled"

Compute the product of the the two last matrices first:

$$a = X(0) + \underline{W^0 X(2)}$$

$$b = X(1) + \underline{W^0 X(3)}$$

$$c = X(0) + W^2 X(2) = X(0) - \underline{W^0 X(2)}$$

$$d = X(1) + W^2 X(3) = X(1) - \underline{W^0 X(3)}$$

4 ~~2~~ multiplications, 4 additions

Multiply by the first matrix:

$$a + \underline{W^0 b}$$

$$a + W^2 b = a - \underline{W^0 b}$$

$$c + \underline{W^1 d}$$

$$c + W^3 d = c - \underline{W^1 d}$$

~~2~~ 4 multiplications

4 additions

TOTAL NUMBER OF OPERATIONS

8 ~~4~~ multiplications

8 additions.