Each examination is marked out of 60.

1. June Examination

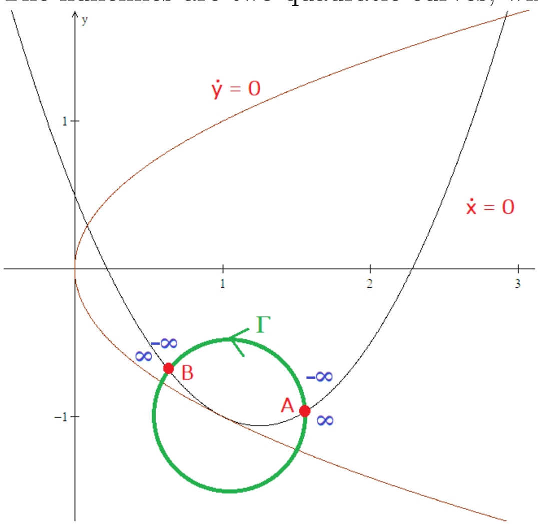
Q1. (20 marks) Consider the following nonlinear inhomogeneous system:

$$\dot{x} = y - x^2 + \frac{5}{2}x - \frac{1}{2}$$
$$\dot{y} = y^2 - x.$$

(i) Verify that there is a fixed point at $(x, y) = (1, -1)$.[2 marks]
(ii) Show that the index at the fixed point $(1, -1)$ is zero. [7 marks]
(iii) Given there are three sectors at this fixed point, sketch with orientation the phase portrait in a neighbourhood of $(1, -1)$. [7 marks]
(iv) What is the local centre manifold at the fixed point $(1, -1)$?[4 marks]

R1. *Examiner's remark: Everybody got two free marks here.* By substitution $(x, y) = (1, -1)$, we find $y - x^2 + \frac{5}{2}x - \frac{1}{2} = 0$ and $y^2 - x = 0$.

The nullclines are two quadratic curves, which look as follows:



At point A, along the green curve Γ, $\dot{y}/\dot{x}$ changes from ∞ to $-\infty$, and at point B, along Γ, $\dot{y}/\dot{x}$ changes from $-\infty$ to ∞. These are the only points along Γ at which $\dot{x} = 0$ and $\dot{y} \neq 0$, and hence the only points at which $|\dot{y}/\dot{x}| = \infty$.

ER: Some candidates drew a finite number of arrows (usually less than seven) without indicating how the arrows had anything to do with integers, much less actually yield a specific integer, the index. Others calculated the number of changes of signs of some undefined quantity, also without indicating what the undefined quantity had anything to do with the index. Both these solutions were deemed unsatisfactory.

The index is given by the total number of full turns of the vector field along a simple closed curve Γ. Taking $\tan(\vartheta) = \mathrm{d}y/\mathrm{d}x$ as usual,

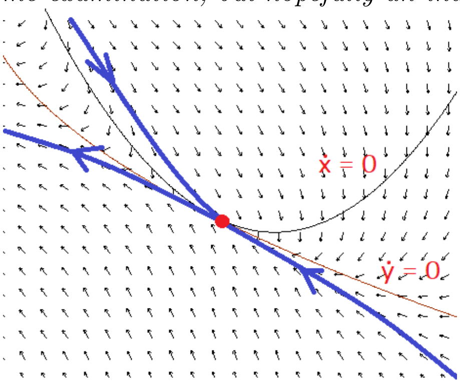
$$\begin{aligned} I((1,-1)) = \frac{1}{2\pi}\oint_\Gamma \mathrm{d}\vartheta &= \frac{1}{2\pi}\oint_\Gamma \frac{1}{1+(\mathrm{d}y/\mathrm{d}x)^2}\mathrm{d}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{1}{2\pi}\oint_\Gamma \frac{1}{1+(\dot{y}/\dot{x})^2}\mathrm{d}\frac{\dot{y}}{\dot{x}} \\ &= \frac{1}{2\pi}\int_{-\infty}^{-\infty} \frac{1}{1+v^2}\mathrm{d}v + \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{1}{1+v^2}\mathrm{d}v = 0, \end{aligned}$$

where the first integral on the last line is taken along Γ from A to B and the second is taken along Γ from B to A. Therefore the index at $(1,-1)$ is zero.

Since the index is nought, by Bendixson's index formula, $1+(e-h)/2=0$, where e is the number of elliptic sectors and h is the number of hyperbolic sectors. This means $e-h=-2$. Since the number of sectors is three, we also have $e+h\le 3$. Since e and h are non-negative integers, there is only one possibility: $(e,h)=(0,2)$.

The remaining sector must be parabolic, and we have a saddle-node in the neighbourhood of $(1,-1)$.

ER: A few candidates did not correctly state Bendixson's index formula, which is curious for a home examination, but hopefully an indication that honesty prevailed.



ER: This last part was longer than I expected upon a revision after I had already sent the question into the exams office. The solution is not difficult, but the calculations are far more tedious than I intended. No candidate completed this calculation entirely and correctly. I made remedies for this in the marking, and was additionally generous with calculation errors.

We shift the fixed point to the origin first to remove the non-homogeneity (this is not strictly necessary, but convenient). With $x'=x-1$ and $y'=y+1$, we arrive at the equations:

$$\dot{x}'=\dot{x}=\frac{1}{2}x'+y'-x'^2, \qquad \dot{y}'=\dot{y}=-x'-2y'+y'^2.$$

We find

$$\begin{pmatrix}1/2 & 1\\ -1 & -2\end{pmatrix}=\begin{pmatrix}-1/2 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}-3/2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-1/2 & -2\\ 1 & 1\end{pmatrix}^{-1}.$$

Since

$$\begin{pmatrix}-1/2 & -2\\ 1 & 1\end{pmatrix}^{-1}=\frac{1}{3}\begin{pmatrix}2 & 4\\ -2 & -1\end{pmatrix}^{-1}$$

Let $w=2x'/3+4y'/3$ and $z=-2x'/3-y'/3$, so $x'=-w/2-2z$ and $y'=w+z$. The system becomes

$$\dot{w}=\frac{2}{3}\dot{x}'+\frac{4}{3}\dot{y}'=-\frac{3}{2}w-\frac{2}{3}(-\frac{w}{2}-2z)^2+\frac{4}{3}(w+z)^2=-\frac{3}{2}w+\frac{7}{6}w^2-\frac{4}{3}z^2+\frac{4}{3}wz,$$
$$\dot{z}=-\frac{2}{3}\dot{x}'-\frac{1}{3}\dot{y}'=-\frac{2}{3}(-\frac{w}{2}-2z)^2-\frac{1}{3}(w+z)^2=-\frac{1}{6}w^2+\frac{7}{3}z^2+\frac{2}{3}wz.$$

Set $F(w,z):=-\frac{1}{6}w^2+\frac{7}{3}z^2+\frac{2}{3}wz$, and $G(w,z):=\frac{7}{6}w^2-\frac{4}{3}z^2+\frac{4}{3}wz$.

By the (local) centre manifold theorem, the equation $w=h(z)$ for the local centre manifold is:

$$\partial_z h(z)\,F(h(z),z)+\frac{3}{2}h(z)-G(h(z),z)=0.$$

ER: A few candidates who got this far asserted that thel linear part of the centre equation was something other than simply zero. This was an obvious mistake.

Since $h(z)$ vanishes to second order, we put in the ansatz $h(z) = az^2 + bz^3 + cz^4 + O(z^5)$, from which we find $h^2(z) = O(z^4)$. We can the PDE out as:

$$\begin{aligned} 0 &= \left(2az + O(z^2)\right)\left(O(z^4) + \frac{7}{3}z^2 + O(z^3)\right) \\ &\quad + \frac{3}{2}\left(az^2 + bz^3 + O(z^4)\right) - \left(O(z^4) - \frac{4}{3}z^2 + \frac{4}{3}\left(az^3 + O(z^4)\right)\right) \end{aligned}$$

(You can verify that the higher order terms written in O-notation only contribute to 4th order terms.)

The only second order terms come from $3h(z)/2$ and $G(h(z), z)$, from which we get $3a/2+4/3 = 0$, and $a = -8/9$. Similarly, the third order terms give us:

$$2az \cdot \frac{7}{3}z^2 + \frac{3}{2}bz^3 - \frac{4}{3}az^3 = 0 \implies b = \frac{160}{81}(\approx 2).$$

The local centre manifold is therefore

$$w = \frac{-8}{9}z^2 + \frac{160}{81}z^3 + O(z^4).$$

Putting $w = 2(x-1)/3 + 4(y+1)/3$ and $z = -2(x-1)/3 - (y+1)/3$ in $w = h(z)$, we can find the equation for the local centre manifold in x and y coordinates to order

$$O((x-1)^3, (y+1)^3, (x-1)^2(y+1)^2).$$

Q2. (15 marks)

Suppose the following system has no fixed points apart from the origin:

$$\begin{aligned} \dot{x} &= -x^5 + x^2y + y^3 + 4x, \\ \dot{y} &= -y^5 - x^3 - xy^2 + 4y. \end{aligned}$$

Determine if system has a non-constant periodic solution lying entirely in the region $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leq 4\}$.

R2. *ER: This question was originally slightly more difficult, requiring a non-constant periodic solution more specifically in* $\{\mathbf{x} \in \mathbb{R}^2 : 1.4 \leq |\mathbf{x}| \leq 2.1\}$. *This is the answer that is presented below.*

Let $r^2 = x^2 + y^2$. The following radial equation can be derived from the system:

$$r\dot{r} = -\left(x^6 + y^6\right) + 4x^2 + 4y^2 = -\left(x^2 + y^2\right)^3 + 3x^2y^2\left(x^2 + y^2\right) + 4\left(x^2 + y^2\right) = -r^6 + 3x^2y^2r^2 + 4r^2.$$

This is not purely in r and the angular argument $\theta = \arctan(y/x)$. The function $(x, y) \mapsto x^2y^2$ is maximised when $x^2 = y^2 = r^2/2$ under the constraint $x^2 + y^2 = r^2$. We can see this by maximizing the $\mathbb{R}_+ \to \mathbb{R}$ map $x^2 \mapsto x^2(r^2 - x^2)$. Therefore

$$r\dot{r} \leq -r^6 + \frac{3}{4}r^6 + 4r^2 = -\frac{1}{4}r^6 + 4r^2.$$

Now $-r^6/4 + 4r^2 < 0$ for $r > (4 \times 4)^{1/4} = 2$, therefore $r\dot{r} < 0$ on $r = 2.1$.

ER: A few candidates asserted that $x^6 + y^6 = r^6$ *with* r *defined as usual. This is a mistake.*

On the other hand,

$$r\dot{r} = -r^6 + 3x^2y^2r^2 + 4r^2 \geq -r^6 + 4r^2,$$

and $-r^6 + 4r^2 > 0$ whenever $r < 4^{1/4} = \sqrt{2}$.

Since there are no fixed points in the closed annular region $\{\mathbf{x} \in \mathbb{R}^2 : \sqrt{2} \leq |\mathbf{x}| \leq 2.1\}$, in which the ω-limit set of trajectories outside this region must lie, the Poincaré–Bendixson theorem asserts that there is a periodic orbit within the annular region, which lies within the disc $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leq 4\}$.

ER: A few candidates attempted a "Lyapunov" way of solving this. This was initially incorrectly marked as wrong, but some people actually made it work and their marks were revised.

Q3. (10 marks) Consider the following planar system with parameter:

$$\dot{x} = 3x + 4y$$
$$\dot{y} = (3 + \mu)x + 2y.$$

Recall the four types of non-degenerate behaviours for planar linear systems at their fixed points: nodes, foci, saddles, and centres — of which the first two can be stable or unstable.

(i) Calculate the value(s) of μ at which the system changes behaviour. [3 marks]

(ii) Draw phase diagrams with orientations of systems for values of μ between each of these critical values and $\pm\infty$. [4 marks]

(iii) Are any of them a bifurcation a saddle-node bifurcation? Explain. [3 marks]

R3. Write the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 3+\mu & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are

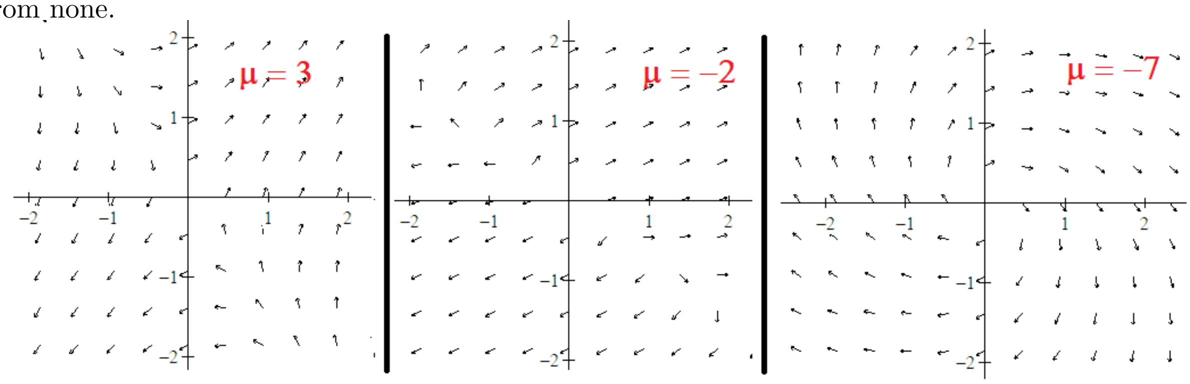
$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}.$$

The system expresses a saddle if $ad - cb < 0$. Putting in $a = 3$, $b = 4$, $c = 3 + \mu$ and $d = 2$, we find that the fixed point $x = 0, y = 0$ is a saddle point if $\mu > -3/2$.

The system expresses a node if $0 < ad + cb < (a + d)^2/4$. This implies $-3/2 > \mu > -49/16$. The node is unstable since $(a + d) = 5 > 0$.

Again, since $a + d > 0$, this collection of systems never expresses a centre. For $\mu < -49/16$, the origin is an unstable focus.

None of these changes in behaviour ar saddle-node bifurcations, at which two fixed points emerge from none.



ER: This question turned out to be the most straightforward.

Q4. (10 marks)

Suppose h is a real-valued function on $[0, T]$ such that

$$\int_0^T |h(t)|\,\mathrm{d}t = \frac{2}{3}.$$

Show that the following equation has a unique solution in the space of real-valued continuous functions $C([0, T])$ equipped with the uniform norm (which makes $C([0, T])$ a complete metric space):

$$f(t) = \int_0^t f(t - s)h(s)\,\mathrm{d}s.$$

R4. *ER: Many candidates did not understand that they had to show existence as well — both "has" and "unique" are important parts of the question — and proceeded to use Gronwall's inequality in differential form to show uniqueness only; some also failed to differentiate the integral equation properly in doing so. Some understood the significance of "has" but nevertheless only could show "unique".*

The uniform norm on $C([0,T])$ is

$$\|f\|_{C([0,T])} = \max_{t\in[0,T]} |f(t)| \, .$$

We show that the map

$$S : f(t) \mapsto \int_0^t f(t-s)h(s)\,\mathrm{d}s$$

is a contraction map in the metric induced by the uniform norm.

Let $f_1, f_2 \in C([0,T])$. We find

$$\begin{aligned}
\max_{t\in[0,T]} |Sf_1(t) - Sf_2(t)| &= \max_{t\in[0,T]} \left| \int_0^t \left(f_1(t-s) - f_2(t-s)\right) h(s)\,\mathrm{d}s \right| \\
&\le \max_{t\in[0,T]} \max_{s\in[0,t]} |f_1(t-s) - f_2(t-s)| \times \max_{t\in[0,T]} \int_0^t |h(s)|\,\mathrm{d}s \\
&= \max_{t\in[0,T]} \max_{s\in[0,t]} |f_1(s) - f_2(s)| \times \max_{t\in[0,T]} \int_0^T |h(s)|\,\mathrm{d}s \\
&\le \max_{t\in[0,T]} |f_1(t) - f_2(t)| \times \int_0^T |h(s)|\,\mathrm{d}s \\
&= \frac{2}{3} \max_{t\in[0,T]} |f_1(t) - f_2(t)| \, .
\end{aligned}$$

Since $2/3 < 1$, the map S is a contraction map on a complete metric space. By the contraction mapping principle, S has a unique fixed point.

Q5. (5 marks) Explain in your own words why the centre manifold theorem is important. Do not copy out the theorem. Explain any associated concepts you introduce.

R5. *ER: Answers were accepted for both the centre manifold theorem or the local centre manifold theorem, the primary point general being that at nonhyperbolic fixed points, the linearised system is not necessarily topologically conjugate to the full dynamics; details were appreciated; other thoughtful responses were also awarded points.*