This examination is marked out of 60.

TMA4165 June 2021 Examination

Q1. (20 marks) Consider the following nonlinear inhomogeneous system:

$$\dot{x} = y - x^2 + \frac{5}{2}x - \frac{1}{2}$$
$$\dot{y} = y^2 - x.$$

(i) Verify that there is a fixed point at $(x, y) = (1, -1)$.[2 marks]

(ii) Show that the index at the fixed point $(1, -1)$ is zero. [7 marks]

(iii) Given there are three sectors at this fixed point, sketch with orientation the phase portrait in a neighbourhood of $(1, -1)$. [7 marks]

(iv) What is the local centre manifold at the fixed point $(1, -1)$?[4 marks]

Q2. (15 marks)

Suppose the following system has no fixed points apart from the origin:

$$\dot{x} = -x^5 + x^2y + y^3 + 4x,$$
$$\dot{y} = -y^5 - x^3 - xy^2 + 4y.$$

Determine if system has a non-constant periodic solution lying entirely in the region $\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leq 4\}$.

Q3. (10 marks) Consider the following planar system with parameter:

$$\dot{x} = 3x + 4y$$
$$\dot{y} = (3 + \mu)x + 2y.$$

Recall the four types of non-degenerate behaviours for planar linear systems at their fixed points: nodes, foci, saddles, and centres — of which the first two can be stable or unstable.

(i) Calculate the value(s) of μ at which the system changes behaviour. [3 marks]

(ii) Draw phase diagrams with orientations of systems for values of μ between each of these critical values and $\pm\infty$. [4 marks]

(iii) Are any of them a bifurcation a saddle-node bifurcation? Explain. [3 marks]

Q4. (10 marks)

Suppose h is a real-valued function on $[0, T]$ such that

$$\int_0^T |h(t)| \, \mathrm{d}t = \frac{2}{3}.$$

Show that the following equation has a unique solution in the space of real-valued continuous functions $C([0, T])$ equipped with the uniform norm (which makes $C([0, T])$ a complete metric space):

$$f(t) = \int_0^t f(t - s)h(s) \, \mathrm{d}s.$$

Q5. (5 marks) Explain in your own words why the centre manifold theorem is important. Do not copy out the theorem. Explain any associated concepts you introduce.