TMA4165 - Exercise set 3 Solutions

Unless otherwise stated the exercises below are from:
D.G. Schaeffer & J.W. Cain: Ordinary Differential Equations: Basic and Beyond.

Exercises for 15-02-2022

Chapter 3:

4 Since $\mathfrak{T}[x](0) = b$ the mean value theorem gives

$$\mathfrak{T}[x](t) - b = t\frac{d}{dt}\mathfrak{T}[x](c) = te^c b + t\cos(t+c)x^2(c)$$

for some c between 0 and t. If $|t| \le \eta$ and $x \in S$ we get

$$\|\mathfrak{T}[x](t) - b\| \le \eta e^{\eta}|b| + \eta(|b| + \delta)^2,$$

Because $|x - b| \le \delta$ implies that $\|x\| \le |b| + \delta$ It follows that $\mathfrak{T}$ maps S into S if

$$\eta e^{\eta}|b| + \eta(|b| + \delta)^2 \le \delta.$$

On the other hand, for any $x, y \in S$ and $t \in [-\eta, \eta]$

$$\mathfrak{T}[x](t) - \mathfrak{T}[y](t) = \int_0^t \cos(s+t)(x^2(s) - y^2(s))ds.$$

Hence

$$\begin{aligned}|\mathfrak{T}[x](t) - \mathfrak{T}[y](t)| &\le \int_0^{|t|} |x^2(s) - y^2(s)|ds \\ &\le \eta\|x^2 - y^2\| \\ &\le \eta\|x+y\|\|x-y\| \\ &\le \eta(\|x\| + \|y\|)\|x-y\| \\ &\le 2\eta(|b| + \delta)\|x-y\|.\end{aligned}$$

Since this holds for any $t \in [-\eta, \eta]$ we obtain

$$\|\mathfrak{T}[x](t) - \mathfrak{T}[y](t)\| \le 2\eta(|b| + \delta)\|x-y\|,$$

which means that $\mathfrak{T}$ is a contraction if

$$2\eta(|b| + \delta) < 1.$$

6 a) We begin by computing a few iterates before making a guess:

$$\begin{aligned}x_1(t) &= x_0(t) + \int_0^t (-sx_0(s))\,ds \\ &= 2 - 2\int_0^t s ds \\ &= 2 - t^2\end{aligned}$$

$$\begin{aligned}x_2(t) &= x_0(t) + \int_0^t (-sx_1(s))\,ds \\ &= 2 - \int_0^t \left(2s - s^3\right) ds \\ &= 2 - t^2 + \frac{t^4}{4}\end{aligned}$$

$$
\begin{aligned}
x_3(t) &= x_0(t) + \int_0^t (-s x_2(s))\, ds \\
&= 2 - \int_0^t \left(2s - s^3 + \frac{s^5}{4}\right) ds \\
&= 2 - t^2 + \frac{t^4}{4} - \frac{t^6}{24} \\
&= 2\left(1 + \left(-\frac{t^2}{2}\right) + \left(-\frac{t^2}{2}\right)^2 \cdot \frac{1}{2!} + \left(-\frac{t^{\cdot 2}}{2}\right)^3 \cdot \frac{1}{3!}\right)
\end{aligned}
$$

This allows us to make the guess that

$$x_n(t) = 2\sum_{k=0}^{n} \left(-\frac{t^2}{2}\right)^n \frac{1}{n!}.$$

We can prove this by induction. It is clearly true for $n = 0$ (and $n = 1, 2, 3$). Moreover if it is true for n then

$$
\begin{aligned}
x_{n+1}(t) &= x_0(t) + \int_0^t (-s x_n(s)) ds \\
&= 2 - 2\int_k^t \sum_{=0}^{n} s\left(-\frac{s^2}{2}\right)^k \cdot \frac{1}{k!} ds \\
&= 2 - 2\sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \cdot \frac{1}{k!} \int_0^t s^{2k+1} ds \\
&= 2 - 2\sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \cdot \frac{1}{k!} \frac{t^{2k+2}}{2k+2} \\
&= 2 + 2\sum_{k=0}^{n} \left(-\frac{t^2}{2}\right)^{k+1} \cdot \frac{1}{(k+1)!} \\
&= 2\sum_{k=0}^{n+1} \left(-\frac{t^2}{2}\right)^k \cdot \frac{1}{k!}.
\end{aligned}
$$

Hence it is true for all n by induction. If we set $u = \left(-\frac{t^2}{2}\right)$ then it is easily seen that

$$x(t) = \lim_{n\to\infty} x_n(t) = 2\sum_{k=0}^{\infty} \frac{u^k}{k!} = 2e^u = 2e^{-\frac{t^2}{2}}$$

b) Now we only have to check that indeed $x(t) = 2e^{-\frac{t^2}{2}}$ is a solution:

$$
\begin{aligned}
\frac{d}{dt}x(t) &= \frac{d}{dt}2e^{-\frac{t^2}{2}} = -t2e^{-\frac{t^2}{2}} = -tx(t) \\
x(0) &= 2e^{-\frac{0^2}{2}} = 2.
\end{aligned}
$$

8 a) We set $h(t) = g(t) + \frac{B}{K}$. Then

$$
\begin{aligned}
h(t) &= g(t) + \frac{B}{K} \\
&\le C + \frac{B}{K} + Bt + K\int_0^t g(s)\, ds \\
&= C + \frac{B}{K} + Bt + K\int_0^t \left(h(s) - \frac{B}{K}\right) ds \\
&= C + \frac{B}{K} + K\int_0^t h(s)\, ds
\end{aligned}
$$

Applying Grönwall's lemma gives

$$h(t) \leq \left(C + \frac{B}{K}\right) e^{Kt}$$

which is equivalent to

$$g(t) \leq Ce^{Kt} + \frac{B\left(e^{Kt} - 1\right)}{K}$$

b) Set $h(t) = g(t) + Ae^{Mt}$ for some constant A to be chosen later. Then

$$\begin{aligned} h(t) &= g(t) + Ae^{Mt} \\ &\leq C(e^{Mt} - 1) + K \int_0^t (h(s) - Ae^{Mt})ds + Ae^{Mt} \\ &\leq C(e^{Mt} - 1) + \left(A - \frac{KA}{M}\right) e^{Mt} + \frac{KA}{M} + K \int_0^t h(s)ds. \end{aligned}$$

Now we want to pick A such that

$$C + A - \frac{KA}{M} = 0$$

which is equivalent to

$$A = \frac{C}{K/M - 1}.$$

This choice of A leaves us with

$$h(t) \leq A + K \int_0^t h(s)ds.$$

Applying Grönwall's lemma to $h(t)$ gives

$$h(t) \leq Ae^{Kt}$$

or equivalently

$$g(t) \leq A(e^{Kt} - e^{Mt}) = \frac{C}{K/M - 1}(e^{Kt} - e^{Mt}).$$

12 Since $(r, \theta) = (1, \pi/2)$ and $(r, \theta) = (1, -\pi/2)$ correspond to the same point in the xy-plane, but $\sin(\pi/2) = 1$ and $\sin(-\pi/2) = -1$, the function cannot be continuous without the cut. On the other hand, with the cut every sufficiently small connected open set in $\mathcal{K}$ can be mapped Bijectively to a connected set in the xy-plane. Consider now

$$\begin{aligned} \frac{|f(P_+) - f(P_-)|}{|P_+ - P_-|} &= \frac{|\sin((\pi - \epsilon)/2) - \sin((\epsilon - \pi)/2)|}{|(\cos(\pi - \epsilon), \sin(\pi - \epsilon)) - (\cos(\pi - \epsilon), -\sin(\pi - \epsilon))|} \\ &= \frac{2\sin((\pi - \epsilon)/2)}{2\sin(\pi - \epsilon)} \\ &= \frac{\sin(\pi/2)\cos(\epsilon/2) - \cos(\pi/2)\sin(\epsilon/2)}{\sin(\pi)\cos(\epsilon) - \cos(\pi)\sin(\epsilon)} \\ &= \frac{\cos(\epsilon/2)}{\sin(\epsilon)} \simeq \frac{1}{\epsilon}, \end{aligned}$$

where the approximate identity holds for small ϵ because

$$\lim_{\epsilon \to 0} \epsilon \frac{\cos(\epsilon/2)}{\sin(\epsilon)} = 1.$$

Chapter 4:

1 a) Since $\mathbf{x}_* : (\alpha_*, \beta_*) \to \mathcal{U}$ it follows that $\mathbf{x}_*(t) \in \mathcal{U}$ for all $t \in (\alpha_*, \beta_*)$. Hence we must have $\lim_{t\to\beta_*} \mathbf{x}_*(t) = x \in \bar{\mathcal{U}}$, that is $x \in \partial\mathcal{U}$ or $x \in \text{int}(\mathcal{U}) = \bar{\mathcal{U}} \setminus \partial U$. We exclude the latter case by contradiction. Assume $x \in \text{int}(\mathcal{U})$, then we can solve

$$(\mathbf{x}^*)' = F(\mathbf{x}^*), \qquad \mathbf{x}^*(\beta_*) = x$$

for all $t \in (\beta_* - \eta, \beta_* + \eta)$. Constructing the function

$$\mathbf{x}(t) = \begin{cases} \mathbf{x}(t) = \mathbf{x}_*(t) & t \in (\alpha_*, \beta_*) \\ \mathbf{x}(t) = \mathbf{x}^*(t) & t \in [\beta^*, \beta^* + \eta) \end{cases}$$

Gives us a solution to

$$\mathbf{x}' = F(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_*(0) \tag{1}$$

on $(\alpha_*, \beta_* + \eta)$. It is clear that this is the case for all times but β_*. However, the function $\mathbf{x}$ is clearly continuous because

$$\lim_{t\to\beta_*^+} \mathbf{x}(t) = \lim_{t\to\beta_*^+} \mathbf{x}^*(t) = x = \lim_{t\to\beta_*^-} \mathbf{x}_*(t) = \lim_{t\to\beta_*^-} \mathbf{x}(t),$$

and applying the mean value theorem we obtain

$$F(\mathbf{x}(c)) = \mathbf{x}'(c) = \frac{\mathbf{x}(\beta_* + h) - x}{h} = \frac{\mathbf{x}(\beta_* + h) - \mathbf{x}(\beta_*)}{h}$$

for any $h \neq 0$ and some c (depending on h) between β_* and β_*+h. Thus the right derivative is

$$\lim_{h\to 0^+} \frac{\mathbf{x}(\beta_* + h) - \mathbf{x}(\beta_*)}{h} = \lim_{c\to\beta_*^+} F(\mathbf{x}(c)) = \lim_{c\to\beta_*^+} F(\mathbf{x}^*(c)) = F(x)$$

and the left derivative is

$$\lim_{h\to 0^-} \frac{\mathbf{x}(\beta_* + h) - \mathbf{x}(\beta_*)}{h} = \lim_{c\to\beta_*^-} F(\mathbf{x}(c)) = \lim_{c\to\beta_*^-} F(\mathbf{x}^*(c)) = F(x),$$

which means $\mathbf{x}$ both is C^1 and solves the eq. (1) on $(\alpha_*, \beta_* + \eta)$.

b) A possible example is $f(x) = \sqrt{1 - x^2}$ which is locally lipschitz as a function $f : (-1, 1) \to \mathbb{R}$. The IVP

$$x' = \sqrt{1 - x^2}, \qquad x(0) = 0,$$

has maximal solution, $x : (-\pi/2, \pi/2) \to (-1, 1)$ given by $x(t) = \sin(t)$.