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1.

(i) By multiplying the first equation by x_2 and the second by x_1 and subtracting the first equation from the first, we see that at a fixed point, we must have $x_1^2 + x_2^2 = 0$, which implies $x_1 = x_2 = 0$. The final equation then gives us $y = 1$ or $y = 0$ at a fixed point. This gives us the the two fixed points

$$p_1 = (0, 0, 0)^\top, \qquad p_2 = (0, 0, 1)^\top.$$

The linearization is governed by

$$\mathrm{D}f = \begin{pmatrix} y & -1 & x_1 \\ 1 & y & x_2 \\ -2x_1 & -2x_2 & 2y-1 \end{pmatrix},$$

which can be evaluated at the fixed points to yield

$$\mathrm{D}f\Big|_{p_1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathrm{D}f\Big|_{p_2} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix $\mathrm{D}f|_{p_1}$ has eigenvalues $\lambda = \pm i, -1$, and $\mathrm{D}f|_{p_2}$ has eigenvalues $\lambda = 1, 1 \pm i$.

That means that p_2 is a hyperbolic fixed point in the neighbourhood of which the behaviour of the system is completely determined by the Hartman-Grobman theorem. In this case, p_2 is an unstable fixed point.

We focus on p_1. We have

$$\mathbf{C} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{P} = -1.$$

We can write the system as

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 y = -x_2 + F_1(x_1, x_2, y) \\ \dot{x}_2 &= x_1 + x_2 y = x_1 + F_2(x_1, x_2, y) \\ \dot{y} &= -y - x_1^2 - x_2^2 + y^2 = -y + G(x_1, x_2, y). \end{aligned}$$

Using the ansatz

$$h(x_1, x_2) = a x_1^2 + b x_1 x_2 + c x_2^2 + O(|\mathbf{x}|^3)$$

(if you attempt the first order ansatz, you will find $h = O(|\mathbf{x}|^2)$), so that

$$\mathrm{D}h(x_1, x_2) = \begin{pmatrix} 2a x_1 + b x_2 \\ b x_1 + 2c x_2 \end{pmatrix} + O(|\mathbf{x}|^2). \tag{1}$$

From the centre manifold theorem we see that we require

$$0 = \mathrm{D}h(x_1, x_2) \cdot \left(\mathbf{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} F_1(x_1, x_2, h(x_1, x_2)) \\ F_2(x_1, x_2, h(x_1, x_2)) \end{pmatrix} \right) - \mathbf{P} h(x_1, x_2) - G(x_1, x_2, h(x_1, x_2)).$$

Now $\mathrm{D}h \cdot (F_1(x_1, x_2, h(x_1, x_2)), F_2(x_1, x_2, h(x_1, x_2)))^\top = O(|\mathbf{x}|^4)$, so we can neglect to calculate those terms. Similarly, $G(x_1, x_2, h(x_1, x_2)) = -x_1^2 - x_2^2 + O(|\mathbf{x}|^4)$. This leaves us with

$$\begin{aligned} 0 &= \mathrm{D}h(x_1, x_2) \cdot \mathbf{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \mathbf{P} h(x_1, x_2) - x_1^2 - x_2^2 + O(|\mathbf{x}|^4) \\ &= (3a + 1) x_1^2 + 3b x_1 x_2 + (3c + 1) x_2^2 + O(|\mathbf{x}|^3). \end{aligned}$$

Since this equation has to be satisfied for any sufficiently small (x_1, x_2), we find that $a = c = -1/3$, and $b = 0$. Hence

$$h(x_1, x_2) = \frac{-1}{3}x_1^2 - \frac{1}{3}x_2^2 + O(|\mathbf{x}|^3), \qquad F(x_1, x_2, h(x_1, x_2)) = -\frac{1}{3}\begin{pmatrix} x_1^3 + x_1 x_2^2 \\ x_1^2 x_2 + x_2^3 \end{pmatrix}.$$

The centre manifold theorem then suggests that on the centre manifold, we locally and qualitatively expect the behaviour

$$\begin{aligned} \dot{x}_1 &= -x_2 - \frac{1}{3}x_1(x_1^2 + x_2^2) \\ \dot{x}_1 &= x_1 - \frac{1}{3}x_2(x_1^2 + x_2^2). \end{aligned}$$

Multiplying $2x_1$ to the first equation and $2x_2$ to the second, we can add them up and use the polar change-of-variables $r^2 = x_1^2 + x_2^2$ on the $x_1 - x_2$ plane to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = -\frac{1}{3}r^4 \le 0.$$

Therefore one expects a stable spiral near the origin on the $x_1 - x_2$ plane.

(ii) Again linear analysis will show that we should focus on "centre" behaviour at the origin.

Here the equations are

$$\begin{aligned} \dot{x}_1 &= x_1 y - x_1 x_2^2 = F_1(x_1, x_2, y) \\ \dot{x}_2 &= -2x_1^2 x_2^2 - x_1^4 + y^2 = F_2(x_1, x_2, y) \\ \dot{y} &= -y + x_1^2 + x_2^2 = -y + G(x_1, x_2, y), \end{aligned}$$

This means that $\mathbf{C}$ is the 2×2 zero matrix and $\mathbf{P} = -1$. Also, G does not in fact depend on y.

We expect the lowest possible order ansatz to be

$$h(x_1, x_2) = ax_1^2 + bx_1 x_2 + cx_2^2$$

because the lowest order term without y in the first and second equations are $O(|\mathbf{x}|^3)$ so we do not have even to attempt a first order ansatz and find that $h = O(|\mathbf{x}|^2)$ as in part (i) of this question.

We find $\mathrm{D}h(x_1, x_2)$ exactly as in (1) foregoing. From the centre manifold theorem, we require

$$0 = \mathrm{D}h(x_1, x_2) \cdot \left(\mathbf{C}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} F_1(x_1, x_2, h(x_1, x_2)) \\ F_2(x_1, x_2, h(x_1, x_2)) \end{pmatrix} \right) - \mathbf{P}h(x_1, x_2) - G(x_1, x_2, h(x_1, x_2)).$$

We can calculate and find that

$$\mathrm{D}h(x_1, x_2) \cdot \begin{pmatrix} F_1(x_1, x_2, h(x_1, x_2)) \\ F_2(x_1, x_2, h(x_1, x_2)) \end{pmatrix} = ax_1^3 + bx_1^2 x_2 + (c-1)x_1 x_2^2 + O(|\mathbf{x}|^4) = O(|\mathbf{x}|^3).$$

Therefore we require that for sufficiently small $|\mathbf{x}|$,

$$0 = (ax_1^2 + bx_1 x_2 + cx_2^2) - x_1^2 + x_2^2 + O(|\mathbf{x}|^3),$$

and this implies $b = 0$, $a = c = 1$, and $h(x_1, x_2) = x_1^2 + x_2^2$.

The centre manifold theorem then suggest that we should expect the dynamics on the centre manifold locally to be governed by

$$\begin{aligned} \dot{x}_1 &= x_1^3 \\ \dot{x}_2 &= x_2^4. \end{aligned}$$

These equations are integrable, and we find

$$x_1(t) = \frac{1}{\sqrt{1/x_1^2(0) - 2t}}, \qquad x_2(t) = \frac{1}{\sqrt[3]{1/x_2^3(0) - 3t}}.$$

These can be plotted parametrically for $(x_1(0), x_2(0))$ near the origin and exhibits a saddle-node.

2. Multiplying the first equation by $2x$ and the second equation by $2y$, and using the polar change-of-variables $r^2 = x^2 + y^2$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}x^2 = -2xy + 2x^2(1-r^2)\sin(|1-r^2|^{-1/2})$$
$$\frac{\mathrm{d}}{\mathrm{d}t}y^2 = 2xy + 2y^2(1-r^2)\sin(|1-r^2|^{-1/2}),$$

Adding these together we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}r^2 = 2r^2(1-r^2)\sin(|1-r^2|^{-1/2}).$$

We can check that this equation holds for $r = 1$ as well.

The derivative is nought exactly when $r = 1$ or when

$$\frac{1}{\sqrt{|1-r^2|}} = n\pi, \qquad n \in \mathbb{Z}.$$

Solving for r^2 yields the radii of the limit cycles expected.

Set

$$r_{\pm n}^2 = 1 \mp \frac{1}{n^2\pi^2}, \qquad \in \mathbb{N}.$$

When $R_\pm^2 = r_{\pm n}^2 + \varepsilon$, $n^{-2}\pi^{-2} > \varepsilon > 0$, for some $\varepsilon' > 0$ dependent on n, we can write

$$\frac{1}{\sqrt{|1-R_\pm^2|}} = n\pi \pm \varepsilon'.$$

Likewise when $r_\pm^2 = r_{\pm n}^2 - \varepsilon$, $n^{-2}\pi^{-2} > \varepsilon > 0$, for some $\varepsilon' > 0$ dependent on n, we can write

$$\frac{1}{\sqrt{|1-r_\pm^2|}} = n\pi \mp \varepsilon'.$$

When n is odd, $\operatorname{sgn}(\sin(n\pi \pm \varepsilon')) = \mp 1$. When n is even $\operatorname{sgn}(\sin(n\pi \pm \varepsilon')) = \pm 1$.

Therefore when n is odd,

$$\operatorname{sgn}(\frac{\mathrm{d}r^2}{\mathrm{d}t}\bigg|_{R_\pm}) = \mp 1, \qquad \operatorname{sgn}(\frac{\mathrm{d}r^2}{\mathrm{d}t}\bigg|_{r_\pm}) = \pm 1,$$

and so the limit cycles of radii $r^2 = 1 + 1/(n^2\pi^2)$ for odd n are stable, and the limit cycles of radii $r^2 = 1 - 1/(n^2\pi^2)$ for odd n are unstable.

Similarly, the limit cycles of radii $r^2 = 1 + 1/(n^2\pi^2)$ for even n are unstable, and the limit cycles of radii $r^2 = 1 - 1/(n^2\pi^2)$ for even n are stable.

3. Verifying that the ansatz provided is indeed a solution is a simple matter of substituting $(x(t), y(t))^\top = (2\cos(2t), \sin(2t))^\top$ into the equations provided and checking that the initial conditions are satisfied.

This cycle, which we shall call γ, has period $T = \pi$. For stability, as this is on $\mathbb{R}^2$, we need only check the sign of the integral

$$\int_0^\pi \nabla \cdot f(\gamma(t))\ \mathrm{d}t,$$

where

$$f(x,y) = \begin{pmatrix} -4y + x(1 - x^2/4 - y^2) \\ x + y(1 - x^2/4 - y^2) \end{pmatrix}, \qquad (\nabla \cdot f)(x,y) = (1-x^2/4-y^2) - x^2/2 + (1-x^2/4-y^2) - 2y^2.$$

Therefore

$$(\nabla \cdot f)(\gamma(t)) = (\nabla \cdot f)(2\cos(2t), \sin(2t)) = -2.$$

We easily find that

$$\int_0^\pi \nabla \cdot f(\gamma(t))\ \mathrm{d}t = \int_0^\pi (-2)\ \mathrm{d}t < 0.$$

This implies the stability of the periodic orbit on $\mathbb{R}^2$.

4. Again we multiply the first equation by $2x$ and the second equation by $2y$, and add them together to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t} r^2 = r^2(r^4 - 3r^2 + 1) = r^2(r^2 - \alpha)(r^2 - \beta),$$

where

$$\alpha = \frac{3 + \sqrt{5}}{2}, \qquad \beta = \frac{3 - \sqrt{5}}{2}.$$

Since $\beta < 1 < \alpha$, we find that $\mathrm{d}(r^2)/\mathrm{d}t < 0$ on $\{r = 1\}$, and as $r > 0$, $\dot{r} < 0$ also.

Similarly, $\beta < \alpha < 2$ implies $\dot{r} > 0$ on $\{r = 2\}$.

The intermediate value theorem applied to $\dot{r}$, which is continuous, suggests that there is a value of r between 1 and 2 for which $\dot{r} = 0$.

Near the origin, $r < \beta < \alpha$, therefore $\mathrm{d}(r^2)/\mathrm{d}t > 0$, and it is unstable. To see that it is a focus, we find the equation for the argument also:

$$\dot{\vartheta} = \frac{1}{r^2}(x\dot{y} - y\dot{x}) = 1 > 0,$$

away from the origin. Since r does not reach 0 in finite time as $t \to -\infty$, it must be that as $t \to -\infty$, $\theta \to \infty$.