1. Let $x \in A \cup B \subseteq X$. Then $f(x) \subseteq f(A)$ or $f(x) \subseteq f(B)$. Therefore $f(A \cup B) \subseteq f(A) \cup f(B)$. Conversely suppose $y \in f(A) \cup f(B)$. Then there is an $x \in A \cup B$ for which $f(x) = y$. Therefore $f(A) \cup f(B) \subseteq f(A \cup B)$.

Let $x \in f^{-1}(U) \cup f^{-1}(V)$. Then $f(x) \in U$ or $f(x) \in V$, and $x \in f^{-1}(U \cup V)$. Therefore $f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V)$. Conversely suppose that $x \in f^{-1}(U \cup V)$, then $f(x) \in U \cup V$, then $f(x) \in U$ or $f(x) \in V$, so $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Therefore $f^{-1}(U \cup V) \subseteq f^{-1}(U) \cup f^{-1}(V)$.

Let $x \in f^{-1}(U) \cap f^{-1}(V)$ Then $f(x) \in U$ and $f(x) \in V$. Therefore $x \in f^{-1}(U \cap V)$, and $f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U \cap V)$. Conversely if $x \in f^{-1}(U \cap V)$, then $f(x) \in U$ or $f(x) \in V$, so $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Therefore $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

Suppose now we try to carry out the same argument for $f(A \cap B)$. If $x \in A \cap B$, $f(x) \in f(A)$ or $x \in f(B)$. Therefore $f(A \cap B) \subseteq f(A \cap B)$. But if $y \in f(A) \cap f(B)$, then all we know is that $f^{-1}(y) \subseteq A \cup B$ because the function may be many-to-one.

In fact we can consider the example of $f(x) = \sin(x)$ and $A = [0, 2\pi]$, $B = [4\pi, 6\pi]$. Here $A \cap B = \varnothing$, but $f(A) \cap f(B) = [-1, 1]$.

2. The fixed point is $\mathbf{x}_0 = (0, 0, 0)$. The linearised system has flux given by

$$\nabla f|_{\mathbf{x}_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvectors spanning the stable subspace E^s are $(1, 0, 0)^\top$ and $(0, 1, 0)^\top$.

The eigenvector spanning the unstable subspace E^c is $(0, 0, 1)^\top$.

We can integrate the full system by hand to find the solution with initial condition $\mathbf{x}(0) = \mathbf{y} = (y_1, y_2, y_3)^\top$.

This yields (by Duhamel's formula or some other method):

$x_1(t) = y_1 e^{-t}$

$x_2(t) = y_2 e^{-t} + y_1^2 \left(e^{-t} - e^{-2t} \right)$

$x_3(t) = y_3 e^t + \frac{1}{3} y_2^2 \left(e^t - e^{-2t} \right) + 2 y_1^2 y_2 \left(\frac{1}{2} e^t - \frac{1}{3} e^{-2t} + \frac{1}{4} e^{-3t} \right) + y_1^4 \left(\frac{1}{30} e^t - \frac{1}{3} e^{-2t} + \frac{1}{2} e^{-3t} - \frac{1}{5} e^{-4t} \right).$

The stable manifold consist of points $\mathbf{y}$ for which $\phi_t(\mathbf{y}) \to \mathbf{x}_0$. From the solution $x_3(t)$, that happens when

$$h_s(\mathbf{y}) := y_3 + \frac{1}{3} y_2^2 + \frac{1}{6} y_1^2 y_2 + \frac{1}{30} y_1^4 = 0.$$

The function h is rank 1 and M_s is codimension 1 level set

$$M_s = \{\mathbf{y} \in U \subseteq \mathbb{R}^3 : h(\mathbf{y}) = 0\},$$

where U is a small neighbourhood of $\mathbf{x}_0$.

The unstable manifold is given by

$$M_u = \{\mathbf{y} \in U \subseteq \mathbb{R}^3 : h_u(\mathbf{y}) := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

The unstable manifold is evidently equal to the unstable subspace already. So we check the tangency condition for the stable manifold. First, $\mathbf{x}_0 \in M_s$. Therefore we check that at $\mathbf{x}_0$, M_s and

E^s have the same normal. The normal of E^s at $\mathbf{x}_0$ is the vector $(0,0,1)^\top$. For M_s, we find

$$\nabla h = \begin{pmatrix} 2y_1^3/15 + y_1 y_2/3 \\ y_1^2/6 + 2y_2/3 \\ 1 \end{pmatrix},$$

so that $\nabla h|_{\mathbf{x}_0} = (0,0,1)^\top$, and E^s is indeed tangent to M_s at $\mathbf{x}_0$.

4. The fixed point of the system is $\mathbf{x}_0 = (0,0,0)^\top$.

The flow of the linearised system around $\mathbf{x}_0$ is

$$\nabla f|_{\mathbf{x}_0} = \begin{pmatrix} -y^2 - 3x^2 & -1 - 2xy & 2z \\ 1 & -3y^2 & 3z^2 \\ -z - 2xz & -z^2 & -x - x^2 - 2yz - 5z^4 \end{pmatrix}\Bigg|_{\mathbf{x}_0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\nabla f|_{\mathbf{x}_0}$ are $\lambda_1 = 0$ and $\lambda_\pm = \pm i$. The corresponding eigenvectors are $\mathbf{v}_1 = (0,0,1)^\top$ and $\mathbf{v}_\pm = (\pm i, 1, 0)^\top$; so $\Re \mathbf{v}_+ = (0,1,0)^\top$ and $\Im \mathbf{v}_+ = (1,0,0)^\top$.

Therefore the linearised system has solutions

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 \mathbf{v}_1 + (C_2 \cos(t) + C_3 \sin(t))\Re \mathbf{v}_+ (C_3 \cos(t) - C_2 \sin(t))\Im \mathbf{v}_+$$

$$= C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \left(\cos(t + \pi/2) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sin(t + \pi/2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$+ C_3 \left(\cos(t) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

where the term C_1 records the height above the $x - y$ plane, and the remaining terms are two circles, $1/4$ out of phase.

Solutions do not tend to the fixed point $\mathbf{x}_0 = (0,0,0)^\top$ as $t \to \infty$, and shows stability but not asymptotic stability of $\mathbf{x}_0$ for the linearised system.

Asymptotic stability of the full sstem can be see from the suggested Lyapunov function $V(x,y,z) = x^2 + y^2 + z^2$. We find that

$$\nabla V \cdot f = 2x\left(-y - xy^2 + z^2 - x^3\right) + 2y\left(x - y^3 + z^3\right) + 2z\left(-xz - x^2 z - yz^2 - z^5\right)$$
$$= -2x^2 y^2 - 2y^4 - 2x^4 - 2x^2 z^2 - 2z^6,$$

which is strictly negative in $U \backslash (0,0,0)$ for any neighbourhood U of the origin. By Lyapunov's theorem, the origin is an asymptotically stable fixed point of the full system.

5. The nullclines of the Lorenz system are

$$0 = \sigma(y - x),$$
$$0 = \rho x - y - xz,$$
$$0 = -\beta z + xy.$$

The fixed points are the intersection of the nullclines. These are at $x = y$, $\rho - z = y/x(= 1$ if $x \neq 0)$, and finally $z = xy/\beta = x^2/\beta$. That is,

$$x = \pm\sqrt{(\rho - 1)\beta} \qquad \text{or} \qquad x = 0.$$

Set $\alpha := \sqrt{(\rho - 1)\beta}$. The fixed points are at

$$p_1 = (0,0,0)^\top, \quad p_2 = (\alpha, \alpha, \rho - 1)^\top, \quad p_3 = (-\alpha, -\alpha, \rho - 1)^\top,$$

with p_2 and p_3 being fixed points only if $\rho - 1 > 0$.

The linearised system has a flux around the origin given by

$$\nabla f|_{p_1} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}\Bigg|_{p_1} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

Its eigenvalues are given by the characteristic equation

$$0 = \lambda^3 + (\sigma + 1 + \beta)\lambda^2 + (\sigma + \beta + \beta\sigma - \rho\sigma)\lambda + \beta\sigma - \beta\rho\sigma.$$

By inspection $\lambda_1 = \beta$ is a root. Factoring out $(\lambda - \beta)$ from the left hand side of the above, we find the remaining eigenvalues

$$\lambda_\pm = \frac{-(\sigma + 1) \pm \sqrt{(\sigma + 1)^2 + 34\sigma(\rho - 1)}}{2}.$$

Therefore the system is stable around the origin if $(\sigma + 1)^2 < 4\sigma(1 - \rho)$ as $\beta, \sigma > 0$. The system is unstable with the inequality reversed.

6. Take $q_1 = x, q_2 = y$, and $p_1 = \dot{x}$ and $p_2 = \dot{y}$. We postulate that

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

for some Hamiltonian function H, and seek the properties of this Hamiltonian function, checking that it does not lead to contradictions. From the Hamiltonian equations above and the equation of $\ddot{x}$ and $\ddot{y}$ assumed, we derive

$$p_i = \frac{\partial H}{\partial p_i}, \quad \frac{q_i}{\left(q_1^2 + q_2^2\right)^{3/2}} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2.$$

Therefore

$$H = \left(q_1^2 + q_2^2\right)^{-1/2} + \frac{p_1^2 + p_2^2}{2}$$

is a suitable Hamiltonian function by which the system in Q6 can be cast into Hamiltonian form.

The orthogonal system of the Hamiltonian system is

$$\dot{q}_i = \frac{\partial H}{\mathrm{d}q_i}, \quad \dot{p}_i = \frac{\partial H}{\partial p_i} \quad \Longrightarrow \quad \dot{q}_i = \frac{q_i}{\left(q_1^2 + q_2^2\right)^{3/2}}, \quad \dot{p}_i = p_i.$$

Substituting in the equations for x and y, we therefore find that the orthogonal system is

$$\ddot{x} = \frac{x}{(x^2 + y^2)^{3/2}}, \quad \ddot{y} = \frac{y}{(x^2 + y^2)^{3/2}}.$$