TMA4165: Sheet II Solutions

1. See June 2018 examination solutions Q7

2. Since $(x-y)(\arctan(x)-\arctan(y)) > 0$ for $x \neq y$ (i.e., the factors have the same sign), it holds that

$$|T(x)-T(y)| = |x-y-\arctan(x)-\arctan(y)| < |x-y|.$$

This does not contradict the contraction mapping principle because even though the space $\mathbb{R}$ is Banach under the absolute value, the map $T : \mathbb{R} \to \mathbb{R}$ is not a contraction, which requires a Lipschitz constant strictly less than one.

3. First let us assume that $u'(0) = K$ is a bounded constant that can be determined from the boundary data for u. Integrating the equation, we find

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = \int_0^x \lambda \sin(u(y)) - f(y)\,\mathrm{d}y + K, \qquad x \in [0,1].$$

Integrating once again, we can use the boundary condition $u(0) = 0$ and find

$$u(x) = \int_0^x \int_0^z \lambda \sin(u(y))\,\mathrm{d}y\,\mathrm{d}z - \int_0^x \int_0^z f(y)\,\mathrm{d}y\,\mathrm{d}z + Kx.$$

If $u \in C([0,1])$ satisfies the integral equation above, it is evidently also in $C^2([0,1])$.

A solution u to the integral equation exists in $C([0,1])$ if the map

$$\mathfrak{T}(u)(x) = \int_0^x \int_0^z \lambda \sin(u(y))\,\mathrm{d}y\,\mathrm{d}z - \int_0^x \int_0^z f(y)\,\mathrm{d}y\,\mathrm{d}z + Kx$$

has a fixed point, which in turn depends on $\mathfrak{T} : C([0,1]) \to C([0,1])$ being a contraction map.

For $u, w \in C([0,1])$, we can estimate as follows:

$$\begin{aligned}
\|\mathfrak{T}(u) - \mathfrak{T}(w)\|_{C([0,1])} &\leq \sup_{x\in[0,1]} \left| \int_0^x \int_0^z \lambda(\sin(u(y)) - \sin(w(y)))\,\mathrm{d}y\,\mathrm{d}z \right| \\
&\leq |\lambda| \sup_{x\in[0,1]} \int_0^x \int_0^z |\sin(u(y)) - \sin(w(y))|\,\mathrm{d}y\,\mathrm{d}z \\
&\leq |\lambda| \|\sin(u) - \sin(w)\|_{C([0,1])}.
\end{aligned}$$

We can conclude that $\mathfrak{T}$ is Lipschitz with Lipschitz constant $|\lambda|$ by the observation that the sine function is differentiable and has derivative bounded by one in absolute values, i.e.,

$$|\sin(u(y)) - \sin(w(y))| \leq |u(y) - w(y)|.$$

Invoking the contraction mapping principle, we see that if $|\lambda| < 1$, then a unique solution to the *initial value problem* with $u(0) = 0$, $u'(0) = K$ exists regardless of K. It remains to show that we can in fact find a $K \in \mathbb{R}$ to ensure $u(1) = 0$. From

$$u(x) = \int_0^x \int_0^z \lambda \sin(u(y))\,\mathrm{d}y\,\mathrm{d}z - \int_0^x \int_0^z f(y)\,\mathrm{d}y\,\mathrm{d}z + Kx.$$

above, we see that we need only set K to be

$$K = -\int_0^1 \int_0^z \lambda \sin(u(y))\,\mathrm{d}y\,\mathrm{d}z + \int_0^1 \int_0^z f(y)\,\mathrm{d}y\,\mathrm{d}z.$$

This ensures well-posedness of the boundary-value problem.

We could also have reduced this to a first-order system by setting $v = \mathrm{d}u/\mathrm{d}x$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}u(x) &= v(x)\\ \frac{\mathrm{d}}{\mathrm{d}x}v(x) &= -\lambda\sin(u(x)) - f(x).\end{aligned}$$

4. We compare $y(t)$ with $z(t) = c/(c-t)$, which also satisfies $z(0) = 1$, and with $c = 1$, is the homogeneous solution.

Taking a derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(y-z) = y^2 - t + \frac{c}{(c-t)^2} = (y^2 - z^2) + \frac{c^2 - c}{(c-t)^2} - t.$$

Now if $(c^2 - c)/(c-t)^2 - t \geq 0$ for $0 \leq t < c$, we can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(y-z) \geq (y+z)(y-z),$$

from which we also have, by an integrating factor (like in Gronwall's inequality)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_0^t y(s)+z(s)\,\mathrm{d}s}(y(t)-z(t))\right) \geq 0.$$

This in turn means $y(t) \geq z(t)$ as $y(0) = z(0)$, and $z(t)$ blows up at $t = c$, so $y(t)$ blows up before $t = c$.

Therefore we seek a lower bound on c for which $(c^2 - c)/(c-t)^2 - t \geq 0$ for $0 \leq t < c$. ...I can only get as good as $c > 1.23$ or so...

5. We seek a solution to the fixed point problem for the map

$$\mathfrak{T}(f) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)\,\mathrm{d}y.$$

We know such a solution must exist and be unique in, e.g., $C([-a,a])$, if we can show that $\mathfrak{T}$ is a contraction map under the uniform norm of $C([-a,a])$.

Let therefore $f, g \in C([-a,a])$. We estimate as follows:

$$\begin{aligned}\|\mathfrak{T}(f) - \mathfrak{T}(g)\|_{C([-a,a])} &= \frac{1}{\pi}\sup_{x\in[-a,a]}\left|\int_{-a}^{a}\frac{1}{1+(x-y)^2}(f(y)-g(y))\,\mathrm{d}y\right|\\ &\leq \frac{1}{\pi}\sup_{x\in[-a,a]}\int_{-a}^{a}\frac{1}{1+(x-y)^2}|f(y)-g(y)|\,\mathrm{d}y\\ &\leq \frac{1}{\pi}\sup_{x\in[-a,a]}\int_{-a}^{a}\frac{1}{1+(x-y)^2}\,\mathrm{d}y\ \|f-g\|_{C([-a,a])}.\end{aligned}$$

The integral can be further evaluated:

$$\int_{-a}^{a}\frac{1}{1+(x-y)^2}\,\mathrm{d}y = \int_{x+a}^{x-a}\frac{1}{1+r^2}\,\mathrm{d}r = \arctan(x-a) - \arctan(x+a).$$

It can be readily verified (by setting the derivative in x to zero) that the integral is maximized when $x = 0$.

Therefore the Lipschitz constant is $2\arctan(a)/\pi$, which is always less than 1 for $a < \infty$. The contraction mapping principle then provides a unique solution to the fixed-point problem.

As $a \to \infty$, this problem may fail to be well-posed.

To see non-negativity, decompose a solution f into $f = f_+ + f_-$, where $f_+ \geq 0$ and $f_- \leq 0$.

Let

$$g_+(x) := 1 + \int_{-a}^{a} K(x-y)f_+(y)\,\mathrm{d}y, \quad g_-(y) := \int_{-a}^{a} K(x-y)f_-(y)\,\mathrm{d}y,$$

and

$$K(x) := \frac{1}{\pi}\frac{1}{1+x^2}.$$

First we see that $g_+ > 0$. Next, since

$$g_-(x) = \int_{-a}^{a} K(x-y)f_-(y)\,\mathrm{d}y \geq \int_{-a}^{a} K(y)\,\mathrm{d}y \min_y f_-(y) > \alpha \min_y f_-(y),$$

for some $\int_{-a}^{a} K(y)\,\mathrm{d}y < \alpha < 1$, and the last inequality is strict unless $\min_y f_-(y) = 0$ (i.e., $f_- \equiv 0$).

By linearity,

$$g_+(x) + g_-(x) = 1 + \int_{-a}^{a} K(x-y)f(y)\,\mathrm{d}y = f(x).$$

Since $g_+ > 0$, if f is not non-negative it must be that

$$\min_x f(x) \geq \min_x g_-(x) \geq \alpha \min_x f_-(x) = \alpha \min_x f(x) > \min_x f(x),$$

a contradiction.

6. The bound in the theorem statement follows from the Gronwall inequality:

$$\begin{aligned}
|\mathbf{x}^\alpha(t) - \mathbf{y}^\beta(t)| &\leq |\mathbf{x}_0 - \mathbf{y}_0| + \int_0^t |f(\mathbf{x}^\alpha(s),\alpha) - f(\mathbf{y}^\beta(s),\beta)|\,\mathrm{d}s \\
&\leq |\mathbf{x}_0 - \mathbf{y}_0| + \int_0^t |f(\mathbf{x}^\alpha(s),\alpha) - f(\mathbf{y}^\beta(s),\alpha)| + |f(\mathbf{y}^\beta(s),\alpha) - f(\mathbf{y}^\beta(s),\beta)|\,\mathrm{d}s \\
&\leq |\mathbf{x}_0 - \mathbf{y}_0| + \int_0^t K^\alpha |\mathbf{x}^\alpha(s) - \mathbf{y}^\beta(s)|\,\mathrm{d}s + C_f'\omega(|\alpha-\beta|)T,
\end{aligned}$$

where C_f' is a constant depending on f, since the modulus of continuity is uniform in the first argument. The theorem is proven with an application of Gronwall's inequality.