TMA4165: Problem Sheet V

published 15/03/2021

1. Let $M : \mathbb{R} \to \mathrm{GL}(\mathbb{R}^d)$ be an invertible matrix-valued C^1 function. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(\det(M)) = \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t} \log(M)),$$

where $\mathrm{d}\log(M)/\mathrm{d}t$ is interpreted as $M^{-1}\mathrm{d}M/\mathrm{d}t$, or look up its proof on the Internet.

2. Find the index about the critical points in the following diagrams:

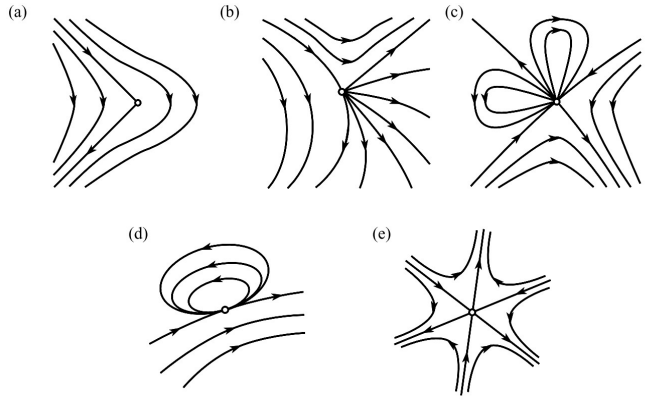

Figure 1. Taken from *Jordan and Smith*

3. Find the index at the critical points of the following systems:

(i) $\dot{x} = 2xy, \;\; \dot{y} = 3x^2 - y^2$;
(ii) $\dot{x} = y^2 - x^4, \;\; \dot{y} = x^3 y$;
(iii) $\dot{x} = x - y, \;\; \dot{y} = x - y^2$.

Find their indices at ∞.

4. Show that for linear planar systems, saddles always have index -1, stable foci always have index 1, centres always have index 1, and stable nodes always have index 1.

5. Show that the following systems have no periodic solutions:

(i) $\dot{x} = y, \;\; \dot{y} = 1 + x^2 - (1 - x)y$;
(ii) $\dot{x} = -(1 - x)^3 + xy^2, \;\; \dot{y} = y + y^3$;
(iii) $\dot{x} = 2xy + x^3, \;\; \dot{y} = -x^2 + y - y^2 + y^3$;
(iv) $\dot{x} = x, \;\; \dot{y} = 1 + x + y^2$;