TMA4165: Problem Sheet IV

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1. Find the approximate flow on the centre manifold of the following systems:

(i)

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1 y\\ \dot{x}_2 &= x_1 + x_2 y\\ \dot{y} &= -y - x_1^2 - x_2^2 + y^2,\end{aligned}$$

which has a stable focus, and

(ii)

$$\begin{aligned}\dot{x}_1 &= x_1 y - x_1 x_2^2\\ \dot{x}_2 &= -2x_1^2 x_2^2 - x_1^4 + y^2\\ \dot{y} &= -y + x_1^2 + x_2^2,\end{aligned}$$

which has a saddle-node in the x_1-x_2 plane.

2. Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)\sin(|1 - x^2 - y^2|^{-1/2})\\ \dot{y} &= x + y(1 - x^2 - y^2)\sin(|1 - x^2 - y^2|^{-1/2}),\end{aligned}$$

where $1 - x^2 + y^2 \neq 0$, completed with the condition that

$$\dot{x} = -y, \qquad \dot{y} = x,$$

on $x^2 + y^2 = 1$.

Show that there is a sequence of limit cycles

$$\Gamma_n^{\pm} = \left\{ (x, y) : \ x^2 + y^2 = 1 \pm \frac{1}{n^2\pi^2} \right\}$$

for $n = 1, 2, \ldots$, accumulating on the unit circle. Comment on the stability of $\Gamma_n^{\pm}$.

3. Show that $(x(t), y(t))^\top = (2\cos(2t), \sin(2t))^\top$ is a periodic solution of

$$\begin{aligned}\dot{x} &= -4y + x(1 - x^2/4 - y^2)\\ \dot{y} &= x + y(1 - x^2/4 - y^2),\end{aligned}$$

with initial conditions $(x_0, y_0)^\top = (2, 0)^\top$.

Show that this periodic solution is in fact a stable limit cycle.

4. Consider the system

$$\begin{aligned}\dot{x} &= -y + x(r^4 - 3r^2 + 1)\\ \dot{y} &= x + y(r^4 - 3r^2 + 1),\end{aligned}$$

where $r = x^2 + y^2$.

Show that $\dot{r} < 0$ on $\{r = 1\}$, and $\dot{r} > 0$ on $\{r = 2\}$. Show that the origin is an unstable focus. Conclude that there is a periodic orbit on the annulus $\{1 < r < 2\}$ and another on $\{0 < r < 1\}$.