

# TMA4165: PROBLEM SHEET III

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- Let  $f : X \rightarrow Y$  be a function on sets. Show that for  $A, B \subseteq X$  and  $U, V \subseteq Y$ ,  $f(A \cup B) = f(A) \cup f(B)$  and  $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ . Does the corresponding statement hold for the intersection in place of the union? If yes prove this, if not, explain the dichotomy.
- Calculate the stable manifold of the following nonlinear system in a small neighbourhood of its fixed point and show that the stable subspace of the linearised system is tangent to the stable manifold at the fixed point:

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= x_3 + x_2^2.\end{aligned}$$

- Find and classify the fixed points of the following systems and draw their phase portraits.
  - $\dot{x} = x - y, \dot{y} = x + y - 2xy$ ;
  - $\dot{x} = 1 - xy, \dot{y} = (x - 1)y$ ;
  - $\dot{x} = (1 + x - 2y)x, \dot{y} = (x - 1)y$ ;
  - $\dot{x} = x - y, \dot{y} = x^2 - 1$ ;
  - $\dot{x} = -6y + 2xy, \dot{y} = y^2 - x^2$ ;
  - $\dot{x} = \sin(x) \cos(y), \dot{y} = \sin(y) \cos(x)$ .
- Use the Lyapunov function  $V(x, y, z) = x^2 + y^2 + z^2$  to show that the origin is an asymptotically stable fixed point of the system

$$\begin{aligned}\dot{x} &= -y - xy^2 + z^2 - x^3 \\ \dot{y} &= x - y^3 + z^3 \\ \dot{z} &= -xz - x^2z - yz^2 - z^5.\end{aligned}$$

Show that the trajectories of the linearized system near  $(x, y, z) = (0, 0, 0)$  lie on circles in planes perpendicular to  $(0, 0, 1)$ , and so the origin is stable but not asymptotically stable for the linearized system.

- For  $\sigma, \rho, \beta > 0$ , the Lorenz equations are:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy.\end{aligned}$$

Determine conditions on the parameters for which the origin is asymptotically stable and conditions for which the origin is unstable.

- Show that the planar two-body problem can be written as a Hamiltonian system with two degrees of freedom in  $\mathbb{R}^4 \setminus \{\mathbf{0}\}$ :

$$\begin{aligned}\ddot{x} &= -\frac{x}{(x^2 + y^2)^{3/2}} \\ \ddot{y} &= -\frac{y}{(x^2 + y^2)^{3/2}}.\end{aligned}$$

Find the gradient system orthogonal to this one in  $\mathbb{R}^4$ .