TMA4165: Problem Sheet II

published: 28/01/2021, revised v2: 31/01/2021

1. (June 2018 examination) Let $x^0(t)$ and $x^\eta(t)$ be solutions to the Cauchy problem

$$0 = \dot{x} - x + \frac{1}{1+r^2}x^2, \qquad x(0) = \frac{1}{2},$$

with $r = 0$ and $r = \eta > 0$, respectively.

Show that for each $r \in \mathbb{R}$, this equation has a unique solution on some interval of existence around $t = 0$, and that the dependence on the parameter r is continuous, in particular, that there is a bounded function $K : t \to \mathbb{R}$ independent of r such that

$$|x^0(t) - x^\eta(t)| \leq K(t)\eta^2.$$

2. Show that

$$T(x) = \frac{\pi}{2} + x - \arctan(x), \qquad x \in \mathbb{R}$$

has no fixed point even though

$$|T(x) - T(y)| < |x - y|$$

for every pair of distinct $x, y \in \mathbb{R}$.

Explain if this contradicts the contraction mapping principle.

3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that the following *boundary value* problem for a *second-order* ODE:

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + \lambda \sin(u(x)) = f(x), \qquad x \in [0, 1]$$
$$u(0) = 0, \qquad u(1) = 0$$

has a unique solution for every sufficiently small λ. Write out the first three Picard iterations. beginning with $u_0 \equiv 0$.

4. (*Perko* Example 2.1) Show that the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = y^2(t) - t, \qquad y(0) = 1$$

exhibits finite-time blow-up, and that this maximal time of existence is less than $t = 1.23$.

5. The following equation for $f : [-a, a] \to \mathbb{R}$ arises in a model for one-dimensional gas dynamics:

$$f(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y)\,\mathrm{d}y, \qquad x \in [-a, a].$$

Show that the integral equation has a unique, non-negative solution for every $0 < a < \infty$. What happens in the asymptotic regime as $a \to \infty$?

6. Exercise 7 in Chapter 4 of *Schaeffer and Cain*, or prove the following, more general version:

Suppose $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is Lipschitz in its first argument and uniformly continouous in its second with modulus of continuouity ω. For solutions $\mathbf{x}^\alpha(t)$ and $\mathbf{y}^\beta(t)$ to

$$\dot{\mathbf{u}} = f(\mathbf{u}, \gamma)$$

on the interval $[0, T]$, respectively with $\gamma = \alpha$ and $\gamma = \beta$, and with initial conditions $\mathbf{x}^\alpha(0) = \mathbf{x}_0$ and $\mathbf{y}^\beta(0) = \mathbf{y}_0$, the following bound holds:

$$|\mathbf{x}^\alpha(t) - \mathbf{y}^\beta(t)| \leq C \min\left((|\mathbf{x}_0 - \mathbf{y}_0| + \omega(|\alpha - \beta|)), |\mathbf{x}_0 - \mathbf{y}_0| e^{C'\omega(|\alpha-\beta|)t}\right),$$

where C and C' are constants depending on T and f, and $\max(\alpha, \beta)$, and on f, respectively.