19. Lecture XIX: Poincaré Map and Stability II

19.1. **Characteristic multipliers, and the Poincaré map.** Last time, after decomposing the fundamental matrix solution Φ of the linearised, non-autonomous system, (which is given by the similar equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = Df|_{\gamma(t)}\Phi(t),)$$

into $\Phi(t) = \mathbf{Q}(t)\exp(\mathbf{B}t)$, where $\mathbf{Q}$ is invertible and B is constant, we said that " it may be guessed that the characteristic multipliers are the eigenvalues of $D\mathbf{\Pi}$ (if one considered $\mathbf{\Pi}$ as a map $\Sigma \subseteq U \to U$ instead of $\Sigma \to \Sigma$)". We shall now unpack, derive, and confirm this speculation.

Let $\phi(t, \mathbf{x})$ be the flow of the full system. The periodic orbit γ with period T containing $\mathbf{x}_0\Sigma$ satisfies $\gamma(t) = \phi(t, \mathbf{x}_0)$. Recall that using the first return time $\tau(\mathbf{x})$ for $\mathbf{x} \in \Sigma$ near $\mathbf{x}_0$, we defined the Poincaré map by

$$\mathbf{\Pi}(\mathbf{x}) := \phi(\tau(\mathbf{x}), \mathbf{x}).$$

We can Taylor expand $\phi(\tau(\mathbf{x}), \mathbf{x})$ around $(T, \mathbf{x}_0)$ to find

$$\begin{aligned}\phi(\tau(\mathbf{x}), \mathbf{x}) &= \phi(T, \mathbf{x}_0) + \partial_t\phi(T, \mathbf{x}_0)(\tau(\mathbf{x}) - T) + D\phi(T, \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + O(|\mathbf{x} - \mathbf{x}_0|^2) \\ &= \phi(T, \mathbf{x}_0) + f(\mathbf{x}_0)D\tau(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + D\phi(T, \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + O(|\mathbf{x} - \mathbf{x}_0|^2).\end{aligned}$$

Therefore the derivative of the Poincaré map at $\mathbf{x}_0$ is

$$D\mathbf{\Pi}|_{\mathbf{x}_0} = D\phi(T, \mathbf{x}_0) = f(\mathbf{x}_0) \cdot D\tau(\mathbf{x}_0) + D\phi(T, \mathbf{x}_0).$$

But $f(\mathbf{x}_0) = \dot{\gamma}(0)$, and in this direction, $\nabla\tau(\mathbf{x}_0)$ does not change. Therefore, $D\mathbf{\Pi}|_{\mathbf{x}_0} = D\phi(T, \mathbf{x}_0)$ (or, defining the Poincaré map to be on the $(d-1)$-dimensional hypersurface, $\tilde{D}\mathbf{\Pi}|_{\mathbf{x}_0} = (D\phi(T, \mathbf{x}_0))_{i,j,=1}^{d-1}$).

Since $\phi_t(t, \mathbf{x})$ is the flow, the matrix $D\phi(t, \mathbf{x})$ satisfies the fundamental matrix solution at $\mathbf{x} = \mathbf{x}_0$, and is a solution with $\Phi(0) = \mathbf{I}_d$. By Floquet's theorem, we can decompose this into

$$D\phi(t, \mathbf{x}_0) = \mathbf{Q}(t)e^{\mathbf{B}t}.$$

Since $H(0, \mathbf{x}_0) = \mathbf{I}_d$, we find $\mathbf{Q}(0) = \mathbf{I}_d$. Since $\mathbf{Q}$ is T-periodic,

$$D\phi(T, \mathbf{x}_0) = e^{\mathbf{B}T}.$$

In each of the directions where the eigenvalue of $\exp(\mathbf{B}T)$ has modulus λ, a point is mapped by that multiple in that direction when it returns, so if $\lambda > 1$, it is mapped further away from $\mathbf{x}_0$ each time it returns, and if $\lambda < 1$ it is mapped closer to $\mathbf{x}_0$ — and hence the orbit draws asymptotically close to the periodic orbit in this (stable) direction. This is the stable manifold theorem for periodic orbits.

In fact, we can say a little more. For hyperbolic orbits:

Theorem 19.1. *Under the hypotheses of the previous theorem, the magnitude of the real parts of the characteristic exponents of the T-periodic orbit Γ are all lower-bounded by $\alpha > 0$. There exists a $K > 0$ such that for each $\mathbf{x} \in M_s(\Gamma)$, there exists an asymptotic phase t_0 such that for all $t \geq 0$,*

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{-\alpha t/T},$$

and for each $\mathbf{x} \in M_u(\Gamma)$, there exists an asymptotic phase t_0 such that for all $t \geq 0$,

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{\alpha t/T}.$$

That is, not only do orbits approach a limit cycle Γ, they also become phase-locked with Γ, exponentially quickly. In a way this is not too surprising, because the flux f governing the dynamics is C^1. These are essentially Gronwall-type estimates.

As with the stable manifold theorem for critical points, there is an associated (weak) centre manifold theorem, asserting the existence of a centre manifold of dimension equal to one less than the number of characteristic exponents with zero real parts. But we should like to revisit the condition of Thm. 18.1,which chararcterised stability with a calculable quantity. This can be generalized to higher dimsions for reasons we have already touched upon in our deductions leading up to the stable manifold theorem for periodic orbits.

19.2. **The fundamental matrix solution.** The point of Floquet's theorem was to resolve for us the problem of dealing with a periodic non-autonomous system. Let us take a closer look at the fundamental matrix solution here.

Theorem 19.2 (Liouville's theorem)**.** *Let γ be a T-periodic orbit of a C^1-first order autonomous system with flux f. Let Φ be the fundamental matrix solution about γ. A necessary but not generally sufficient condition for the orbit γ to be asymptotically stable is that*

$$\log(\det \Phi(t)) = \int_0^T (\nabla \cdot f)(\gamma(t)) \, \mathrm{d}t \leq 0.$$

This theorem implies Thm. 18.1. This follows directly from Jacobi's formula that we mentioned in the last lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log(\det(\Phi))) = \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t}\log(\Phi)),$$

putting in the formula for the fundamental matrix solution into the temporal derivative.

For linear equations, the Wronskian is a fundamental matrix solution, and this follows from Abel's theorem.

The non-sufficiency comes from the fact that we need the eigenvalues of $\int_0^T \mathrm{D}f(\gamma(t)) \, \mathrm{d}t$ all to be negative except the 0 eigenvalue arising from the fact that the Poincaré map maps onto a codimension one surface, whereas the condition stated in the theorem is merely the trace of this quantity. This would have been enough in dimension $d = 2$, where only one eigenvalue is unspecified apart from the 0.