Example 16.3. The system

$$\dot{x} = -y + x(1 - x^2 - y^2), \qquad \dot{y} = x + y(1 - x^2 + y^2), \qquad , \dot{z} = \alpha$$

has an attracting set $\{x^2 + y^2 = 1\}$. We can see this by using the polar coordinates $r^2 = x^2 + y^2$ even though this is a three-dimensional system because the equations are decoupled, the first two from the final.

Example 16.4. The Lorenz system was originally suggested as a model for atmospheric convection in 1963.

We briefly mention the Lorenz system, which has appeared in an exercise some weeks before:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z,\end{aligned}$$

where $\sigma, \rho, \beta > 0$.

Where $\alpha = \sqrt{(\rho - 1)\beta}$, we have seen that the fixed points of this system are at

$$p_1 = (0, 0, 0)^\top, \qquad p_2 = (\alpha, \alpha, \rho - 1)^\top, \qquad p_3 = (-\alpha, -\alpha, \rho - 1)^\top.$$

.

We know that the centre manifold of two dimensional analytic systems can be characterized more or less completely, even if behaviours can differ from linear systems substantially. For appropriate values of σ, ρ and β, (say, $(\sigma, \rho, \beta) = (10, 24.5, 8/3)$ the Lorenz system has a curiously complicated attractor that two dimensions cannot support. This is the case even though one of the equations of the system is linear, and the remaining are also analytic.

The attractor A of this system is made of an infinite number of banched surfaces each of which intersect. Trajectories, being of an autonomous system, do not intersect as they move along A, but there are nevertheless periodic trajectories of arbitrarily large period, uncountably many non-periodic trajectories, and also trajectories that are dense in A. We refer to these attractors as "strange attractors".

17. Lecture XVII: Limit Sets II

17.1. **Limit Cycles.** Now we turn to a fuller discussion of limit cycles but confine ourselves to the plane.

Cycles, or periodic orbits are closed-curve solutions that are not equilibria, a definition exactly in line with our previous usage of the term. Since our systems are autonomous, if the trajectory Γ is a cycle, there is a minimal T independent of $\mathbf{x}_0 \in \Gamma$ for which $\phi_{t+T}(\mathbf{x}_0) = \phi_t(\mathbf{x}_0)$. This minimum T we call the period of the Γ. For centres of linear systems, the period is constant over a family of periodic orbits. This is not so in general. We shall be spending the next few lectures looking at the behaviour of periodic solutions and limit cycles, especially in two dimensions.

A cycle can be itself unstable, stable, and asmptotically stable. To discuss these notions analogously to the way we did for fixed points, we digress briefly to mention that the distance from a point $\mathbf{x}$ to a set E is defined as

$$d(\mathbf{x}, E) := \inf_{\mathbf{y} \in E} |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^d}.$$

Then we say that a cycle Γ is stable if for every $\varepsilon > 0$, there is a neighbourhood U of Γ (an open set U containing Γ) for which $\mathbf{x} \in U$ implies

$$d(\phi_t(\mathbf{x}), \Gamma) < \varepsilon.$$

The cycle Γ is unstable if it is not stable. And analogous to asymptotic stability previously defined, Γ is asymptotically stable if it contains the ω-limit set of every trajectory within a certain neighbourhood U of itself. That is, Γ is stable, and there exists a neighbourhood U of Γ such that $\mathbf{x} \in U$ implies

$$\lim_{t\to\infty} d(\phi_t(\mathbf{x}), \Gamma) = 0.$$

Closed curves that are equilibria of sorts (not an equilibrium in the sense that they sit at the intersection of nullclines!) constitute an important class of limit sets known as LIMIT CYCLES, which we have encountered in the specific context of centre-foci. These are cycles that are the α or ω set of some trajectories. And if the entire cycle is a limit set, then there is also the notion of semistability that can be defined. If there exists a neighbourhood U of Γ for which Γ is the ω-limit set (resp. α-limit set) for every trajectory in/passing through U, then Γ is an ω-LIMIT CYCLE, or a STABLE LIMIT CYCLE (resp. α-LIMIT CYCLE, or a UNSTABLE LIMIT CYCLE). If Γ is an α-limit set for one trajectory and an ω-limit set for another trajectory, then we say that it is a SEMISTABLE LIMIT CYCLE.

Example 17.1. The system

$$\dot{x} = -y + x(1 - z^2 - x^2 - y^2), \qquad \dot{y} = x + y(1 - z^2 - x^2 - y^2), \qquad \dot{z} = 0$$

has an attracting set that is $S^2 \cup \{(0,0,z) : |z| > 1\}$. To see this, use polar coordinates — taking $x = r\cos(\theta)\sin(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$ and $z = r\cos(\varphi)$, so that $r^2 = x^2 + y^2 + z^2$, we find

$$\dot{\overline{r^2}} = 2x\dot{x} + 2y\dot{y} + 2z\dot{z} = -2xy + 2x^2(1 - r^2) + 2xy + 2y^2(1 - r^2) = 2(x^2 + y^2)(1 - r^2).$$

On $\{r < 1\}$, we see that $\dot{r} > 0$, whereas on $\{r > 1\}$, $\dot{r} < 0$ but $\dot{z} = 0$.

The ω-limit set is a stable limit cycle — consider the azimuthal angular frequency:

$$\begin{aligned}
\dot{\theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\arctan(\frac{x}{y}) \\
&= \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \\
&= \frac{1}{r}\left(x^2 + xy(1 - z^2 - r^2) + y^2 - xy(1 - z^2 - y^2)\right) \\
&= 1.
\end{aligned}$$

On $\{r = 1\}$, $\dot{\theta} = 1$.

Cycles, like points, have stable and unstable manifolds. For a cycle Γ, and U a neighbourhood thereof, we define the local stable and unstable manifolds as

$$M_s(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to\infty}{\to} 0,\ \forall t \geq 0,\ \phi_t(\mathbf{x}) \in U\},$$

and

$$M_u(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to-\infty}{\to} 0,\ \forall t \leq 0,\ \phi_t(\mathbf{x}) \in U\},$$

respectively. The global stable and unstable manifolds of a cycle are then

$$\begin{aligned}
W^s(\Gamma) &= \bigcup_{t\leq 0} \phi_t(M_s(\Gamma)) \\
W^u(\Gamma) &= \bigcup_{t\geq 0} \phi_t(M_u(\Gamma)).
\end{aligned}$$

This definition makes the global stable and unstable manifolds invariant under ϕ_t. We shall see that in many ways, we can treat a periodic orbit as a critical point, even though there are important ways that we cannot so do.

17.2. **Some results in $\mathbb{R}^2$.** In $\mathbb{R}^2$, there is a fundamental result about simple closed curves (i.e., closed curves $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(t) = \gamma(T)$ implies $T = 1$ and $t = 0$, or vice versa), known as the Jordan curve theorem:

Theorem 17.1 (Jordan curve theorem)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple cloesd curve. Then $\mathbb{R}^2 \backslash \{\gamma(t) : t \in [0,1]\}$ is a disjoint union of two open sets.*

In fact, simple closed curves are also known as Jordan curves. This might seem like a trivial result, but its proof is not easy. A hint of its depth is that we can characterize the topological dimension of a space by removing objects. One of the fundamental properties distinguishing $\mathbb{R}$ from $\mathbb{R}^d$, $d > 2$, is that by removing a point from $\mathbb{R}$, it becomes disconnected — it is the disjoint union of two open sets.

We call these two disjoint parts the "interior" and the "exterior".

It turns out that a limit cycle has to be approached maximally tangentially. We can think of a approach to a limit cycle as simliar to an approach to a point that sits at the centre of a focus, and we have the following theorem that is analogous to the second part of Thm.13.1 [(in an analytic planar system, if one trajectory spirals, then all do)]:

Theorem 17.2. *If one trajectory in the exterior of a limit cycle Γ of a planar C^1-system has Γ as its ω-limit set (resp. α-limit set), then every trajectory in some exterior neighbourhood of Γ does so also. Moreover any such trajectory spirals towards Γ as $t \to \infty$ (resp. $t \to -\infty$) in the sense that it intersects any line segment normal to Γ at a point on a sequence of times $\{t_n\}$ diverging to infinity (resp. negative infinity).*

In polar terms, from any $\mathbf{y}$ in the interior of Γ, the trajectory $\phi_t(\mathbf{x}_0)$ satisfies

$$d(\phi_t(\mathbf{x}_0), \Gamma) \searrow 0, \qquad \arg(\phi_t(\mathbf{x}_0) - \mathbf{y}) \to \infty,$$

as $t \to \infty$.

Example 17.2. Recall from Dulac's theorem (Thm. 11.3) that analytic planar systems can have at most finitely many limit cycles. We shall look at a non-analytic planar system that exhibits a centre-focus:

$$\dot{x} = -y + x(x^2 + y^2) \sin(\frac{1}{\sqrt{x^2 + y^2}})$$
$$\dot{y} = x + y(x^2 + y^2) \sin(\frac{1}{\sqrt{x^2 + y^2}}).$$

In polar coordinates this becomes:

$$\dot{r} = r^3 \sin(1/r), \qquad \dot{\vartheta} = 1.$$

As we can see, at $r = 1/(n\pi)$, for every $n \in \mathbb{N}$, there is a limit cycle, and between each limit cycle are alternating stable and unstable spirals that turn in the same sense.

Example 17.3. Consider next the Hamiltonian system

$$\dot{x} = y, \qquad \dot{y} = x + x^2.$$

This is a planar Hamiltonian system with $H(x, y) = y^2/2 - x^2/2 - x^3/3$. That is, the solution curves are given by $y^2/2 - x^2/2 - x^3/3 = C$. For $C = 0$, we find the folium of Descartes, and a trajectory that starts and ends at the origin, where locally the system has a saddle critical point. There is a centre at $(-3/2, 0)$ (which we have tools to verify is in fact a centre). The α- and ω- limit sets of this trajectory are both $\{0\}$. This is an example of a homoclinic orbit, which is also a separatrix cycle.

Hamiltonian systems often have cyclic behaviour as energy passes from one mode to another and then back again, being conserved throughout the cycle. Recalling Example 11.1, we call the separatrices of the saddles HETEROCLINIC as they "bend towards" different limit sets.