But the Thm. 13.1 also allows for more diverse behaviours under its second non-spiralling provision. It is in fact possible, as shall be demonstrated in computable examples later, that the plane gets divided into sectors separated much like saddles by separatrices that approach the critical point along definite directions as $t \to \infty$ or as $t \to -\infty$. We denominate three possible types of sectors as being HYPERBOLIC, PARABOLIC, ELLIPTIC according as there is a small enough neighbourhood about the critical point such that each trajectory in the sector not including the separatrices

(i) leaves the neighbourhood as $t \to \pm\infty$, or
(ii) leaves the neighbourhood as $t \to \infty$ and approaches the critical point as $t \to -\infty$, or vice versa, or
(iii) approaches the critical point as $t \to \pm\infty$.

The saddle is then seen to be a critical point with four separatrices and four hyperbolic sectors, and a node is a critical point with one single parabolic sector.

As mentioned at the beginning of this lecture, from the Hartman–Grobman Theorem, or more directly, from Thm.12.2, we know that other sectoring behaviours are not exhibited at hyperbolic fixed points. We also have considered behaviours around critical points for which the linearized system exhibits centres, and Thm. 12.3 ensures that in the analytic case, the full dynamics about these critical points are centres or foci. This leaves critical points with one or two eigenvalues that are zero.

13.2.1. One zero eigenvalue.

We first consider systems with one eigenvalue set to nought. Again, from the theorem on Jordan normal forms (Thm. 4.1), we know that the linearized system is governed by

$$\mathrm{D}f = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

By scaling, we may take $\lambda = 1$, without loss of generality. If we desire that λ correspond to a stable subspace, we simply consider the system backwards in time. This compels us to consider systems of the type:

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= y + Q(x, y), \end{aligned}$$

where P and Q vanish to second order around the *isolated* fixed point $(x_0, y_0)^\top = \mathbf{0}$.

From the implicit function theorem (Thm.8.4), there is a function ϕ such that $y = \phi(x)$ solves $y + Q(x, y) = 0$ in a neighbourhood of $\mathbf{0}$. This is, locally, graph of the nullcline. We can write $\phi(x)$ as

$$\phi(x) = \phi(0) + \phi'(0)x + \cdots,$$

as we are in an analytic setting. Furthermore, since P is analytic and vanishes to second order in a neighbourhood of $\mathbf{0}$, we can write

$$\psi(x) = P(x, \phi(x)) = \sum_{m \geq 2} a_m x^m.$$

It turns out that the lowest order term of ψ — P evaluated on the other nullcline — gives us further information beyond the linearization that can already classify the behaviour of the system in a neighbourhood of the critical point.

First we need to look at some classes of behaviours:

(i) a critical point is a CRITICAL POINT WITH AN ELLIPTIC DOMAIN if it has four separatrices, one elliptic sector, one hyperbolic sector, and two parabolic sectors;
(ii) a critical point is a SADDLE-NODE if it has three separatrices, two hyperbolic sectors, and one parabolic sector; and
(iii) a critical point is a CUSP if it has two separatrices and two hyperbolic sectors.

And we have the following theorem:

Theorem 13.2. *Let $\psi(x) = P(x, \phi(x)) = \sum_{m \geq 2} a_m x^m$ be defined as before in a neighbourhood of the origin for the planar system governed by $P(x, y)$ and $Q(x, y)$, also previously defined. Let ℓ be the smallest integer for which $a_\ell \neq 0$.*

(i) *If $\ell \equiv 1\,(\text{mod}.2)$ and $a_\ell > 0$, then* **0** *is an unstable node,*
(ii) *if $\ell \equiv 1\,(\text{mod}.2)$ and $a_\ell < 0$, then* **0** *is a (topological) saddle, and*
(iii) *if $\ell \equiv 0\,(\text{mod}.2)$, then* **0** *is an saddle-node.*

13.2.2. Two zero eigenvalues with unit geometric multiplicity.

The corresponding theorem becomes considerably more complicated when both eigenvalues are zero, but the dimension of $\ker(\mathrm{D}f)$ is only one. We mention this for completeness.

In this case, it turns out that there is always a change-of co-ordinates that leaves the system in the following normal form:

$$\dot{x} = y$$
$$\dot{y} = a_k x^k + b_n x^n y + O(x^{k+1} + x^{n+1} y + y^2)$$

where the higher order terms are dominated by the first two terms in a neighbourhood of the fixed point **0**.

We define two further parameters:

$$m := [k/2], \qquad \lambda := b_n^2 + (2k+2)a_k.$$

The behaviours exhibited at the fixed point obey the following schematic:

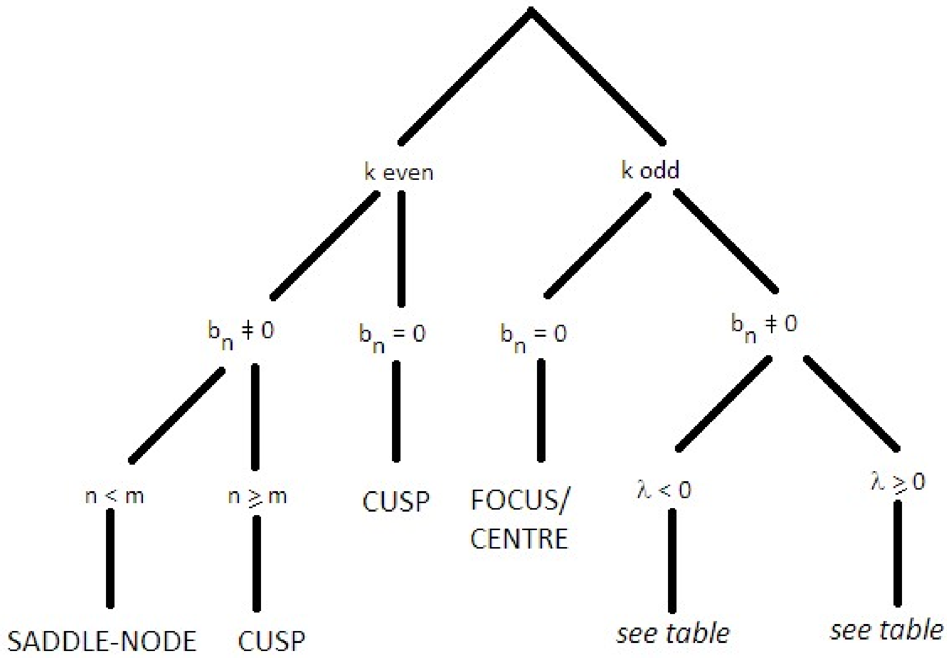

Figure 2. behaviours at critical points

The schematic above must also be supplemented by the following tables:
For $\lambda < 0$,

$b_n \neq 0$, $\lambda < 0$	$n < m$	$n = m$	$n > m$
n even	node	focus/centre	focus/centre
n odd	elliptic domain	focus/centre	focus/centre

and for $\lambda \geq 0$,

$b_n \neq 0$, $\lambda \geq 0$	$n < m$	$n = m$	$n > m$
n even	node	node	focus/centre
n odd	elliptic domain	elliptic domain	focus/centre

.

[There is no expectation that this schematic and its associated tables be committed to memory.]

13.2.3. Two zero eigenvalues with geometric multiplicity of two.

In this case, there are very few general theorems, and the behaviour on the centre manifold can be very complicated. In particular, if P and Q both vanish to order m around $\mathbf{0}$, then the plane is locally split into $(2m+1)$ sectors. We shall look at this case in greater detail in our discussion on index theory.

13.3. **Examples.**

Example 13.1. The system

$$\dot{x} = x^2, \qquad \dot{y} = y,$$

has a linearization governed by

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and so has only one zero eigenvalue. It falls into case (iii) of Thm. 13.2. Therefore we expect a saddle-node.

Example 13.2. The system

$$\dot{x} = y, \qquad \dot{y} = -x^3 + 4xy,$$

has a linearization governed by

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and so has two zero eigenvalue, and $\dim\ker(\mathrm{D}f) = 1$. We find that $k = 3$, $m = 1$, $n = 1$, $a_3 = -1$, $b_1 = 4 \neq 0$, and $\lambda = b_1^2 - (2k+2)a_k = 24 > 0$. Consulting the schematic, we expect a critical point with an elliptic domain.