9. Lecture IX: Stable Manifold and Hartman–Grobman Theorems

9.1. Stable Manifold Theorem. Now we are ready to discuss the ideas of the stable manifold theorem.

Recall that for a linear system, by Thm.4.3, we can decompose phase space about its critical points to $\mathbb{R} = E^s \oplus E^c \oplus E^u$, where

$$\begin{aligned} E^s &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n<0\}} V_n \\ E^c &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n=0\}} V_n \\ E^u &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n>0\}} V_n, \end{aligned}$$

and V_n were the vectors spanning the general eigenspace associated with λ_n. Now if $\mathbf{0}$ is a hyperbolic critical point, by definition, $\mathrm{D}f(\mathbf{0})$ does not have eigenvalues with $\Re\lambda_n = 0$. Therefore, for the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{0})\mathbf{x}(t),$$

we have a decomposition of phase space into $\mathbb{R}^d = E^s \oplus E^u$.

The stable manifold theorem is as follows:

Theorem 9.1 (Stable Manifold Theorem)**.** *Let U be an open subset of $\mathbb{R}^d$ containing the origin. Let $f \in C^1(U;\mathbb{R}^d)$, and let ϕ_t be the flow of the nonlinear system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x}).$$

Suppose that $f(\mathbf{0}) = \mathbf{0}$ and $\mathrm{D}f(\mathbf{0})$ has k eigenvalues (counting multiplicity) with negative real parts and $d-k$ eigenvalues with positive real parts. Then

(i) *there exists a dimension k C^1-manifold M_s tangent to E^s of the linearized system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathrm{D}f(\mathbf{0})\mathbf{x}$$

at $\mathbf{0}$ such that for all $t \geq 0$, $\phi_t(M_s) \subseteq M_s$ and for all $\mathbf{y} \in M_s$,

$$\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0};$$

and

(ii) *there exists a dimension $d-k$ C^1-manifold M_u tangent to E^u of the linearized system at $\mathbf{0}$ such that for all $t \leq 0$, $\phi_t(M_u) \subseteq M_u$ and for all $\mathbf{y} \in M_u$,*

$$\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}.$$

Example 9.1. We look at a nonlinear system that we can solve explicitly:

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= x_3 + x_1^2. \end{aligned}$$

There is one fixed point, which is the origin. The linearization is given by

$$\mathrm{D}f(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so $\mathbf{0}$ is a hyperbolic critical point. The eigenvalues and eigenvectors are readily deducible, and we see that

$$E^s = \{\mathbf{x} : x_2 = 0\} = \text{span}\{(1,0,0)^\top, (0,1,0)^\top\}, \qquad E^u = \{\mathbf{x} : x_1 = x_2 = 0\} = \text{span}\{(0,0,1)^\top\}.$$

The equations can also be integrated by hand, giving

$$\begin{aligned} x_1(t) &= y_1 e^{-t} \\ x_2(t) &= y_2 e^{-t} + y_1^2(e^{-t} - e^{-2t}) \\ x_3(t) &= y_3 e^{t} + y_1^2(e^{t} - e^{-2t})/3, \end{aligned}$$

with $\mathbf{x}(0) = \mathbf{y} = (y_1, y_2, y_3)^\top$.

We see that $\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1^2/3 + y_3 = 0$, and so

$$M_s = \{\mathbf{y} \in \mathbb{R}^3 : y_1^2 + 3y_3 = 0\}.$$

Likewise, $\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1 = y_2 = 0$, and so

$$M_u = \{\mathbf{y} \in \mathbb{R}^3 : y_1 = y_2 = 0\}.$$

It is clear that M_u is tangent to E^u at $\mathbf{0}$ because they coincide entirely. Taking the derivative of $h(y_1, y_2, y_3) = y_1^2 + 3y_3$, for which M_s is the level set at 0, we find

$$\nabla h\Big|_{\mathbf{0}} = (2y_1, 0, 3)^\top|_{(0,0,0)} = (0,0,3),$$

which is indeed perpendicular to E^s, and so S and E^s are tangent at $\mathbf{0}$, as expected.

The way that the Stable Manifold Theorem is usually proven gives us insight into the structure of nonlinear systems. And whilst we shall not be proving the Stable Manifold Theorem, it is of benefit to discuss some elements of its proof. First notice that for a general first order, autonomous nonlinear system, we have the following Taylor's expansion around a hyperbolic critical point $\mathbf{x}_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{x}_0)\mathbf{x}(t) + \mathbf{G}(\mathbf{x}),$$

where $\mathbf{G}$ has zero first derivative at $\mathbf{x}_0$. This means that whilst $\mathbf{G}$ might not be "second-order" in $\mathbf{x}$, for every ε, there is a δ such that if $|\mathbf{x} - \mathbf{x}_0| < \delta$, and $|\mathbf{y} - \mathbf{x}_0| < \delta$,

$$|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})| \leq \varepsilon |\mathbf{x} - \mathbf{y}|.$$

By applying the Jordan Normal Form Theorem (Thm. 4.1), we can assume that $\mathrm{D}f(\mathbf{x}_0)$ is of the form

$$\mathrm{D}f(\mathbf{x}_0) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P is a matrix in Jordan normal form with only eigenvalues of negative real parts, and Q is a matrix in Jordan normal form with only eigenvalues of positive real parts. The linear system can be solved by exponentiating $\mathrm{D}f(\mathbf{x}_0)$ so that where

$$W(t) = \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix},$$

the flow of the linearized system is

$$e^{\mathrm{D}f(\mathbf{x}_0)t} = W(t) + Z(t).$$

Using the Duhamel representation to treat the term $\mathbf{G}$ as an inhomogeneity, we find

$$\mathbf{x}(t) = e^{\mathrm{D}f(\mathbf{x}_0)t}\mathbf{b} + \int_0^t e^{\mathrm{D}f(\mathbf{x}_0)(t-s)}\mathbf{G}(\mathbf{x}(s))\,\mathrm{d}s$$

If we look at solutions that start on what might potentially be the stable manifold, we have solutions of the form

$$\mathbf{x}(t) = W(t)\mathbf{b} + \int_0^t W(t-s)\mathbf{G}(\mathbf{x}(s))\,\mathrm{d}s - \int_t^\infty Z(t-s)\mathbf{G}(\mathbf{x}(s))\,\mathrm{d}s.$$

Now we can take Picard iterations and exploit the signs of the real parts of the eigenvalues to bound W and Z to reach our conclusions via an applicaiton of the contraction mapping theorem on a sufficiently small ball around the fixed point. With these same estimates, it can be shown that the solution to which the Picard approximation tend also satisfies

$$|\mathbf{x}(t;\mathbf{b})| \leq K|\mathbf{b}|e^{-\alpha t}$$

for initial conditions $\mathbf{b}$ sufficiently close to the fixed point $\mathbf{x}_0$ and $\alpha > 0$ chosen as to be smaller in magnitude than all negative eigenvalues of $Df(\mathbf{x}_0)$. A similar procedure can be done for the unstable manifold, except we then run time in the backwards direction using the reversal $t \mapsto -t$.

We can see what form the stable manifold must take from the Duhamel representation by taking $t = 0$. First, taking $t = 0$, we find

$$\mathbf{b} = \mathbf{x}(0;\mathbf{b}) + \int_0^\infty Z(t-s)\mathbf{G}(\mathbf{x}(0;\mathbf{b}))\,\mathrm{d}s.$$

Next, by the form of the solution, it is clear that $\mathbf{x}(t;\mathbf{b})$ is independent of the last $d-k$ coordinates of $\mathbf{b}$. So we are led to the equations defining a manifold:

$$\mathbf{0} = (y_1, y_2, \ldots, y_d)^\top - W(0)\mathbf{x}(0; y_1, \ldots, y_k, 0, \ldots, 0) - \int_0^\infty Z(t-s)\mathbf{G}(\mathbf{x}(0; y_1, \ldots, y_k, 0, \ldots, 0))\,\mathrm{d}s.$$

The first k equations yield no information as they only say $0 = 0 - 0$. The final $d-k$ equations are level set (at $\mathbf{0} \in \mathbb{R}^{d-k}$) of the map

$$\mathbf{y} \in \mathbb{R}^d \mapsto (y_{k+1}, \ldots y_k)^\top - \int_0^\infty Z(t-s)\mathbf{G}(\mathbf{x}(0; y_1, \ldots, y_k, 0, \ldots, 0))\,\mathrm{d}s,$$

which is a manifold of codimension $d-k$, as sought. It is more effort to show that solutions not beginning on this manifold do not tend to the fixed point $\mathbf{x}_0$ as $t \to \infty$.

The Stable Manifold Theorem only defines M_s and M_u on a small neighbourhood of the hyperbolic critical point. To supplement their definition in the theorem we also introduce the GLOBAL STABLE AND UNSTABLE MANIFOLDS at $\mathbf{0}$ if it is a hyperbolic fixed point:

$$W^s(\mathbf{0}) = \bigcup_{t \leq 0} \phi_t(M_s)$$
$$W^u(\mathbf{0}) = \bigcup_{t \geq 0} \phi_t(M_u).$$

These may not be manifolds in the sense we have defined, or in the more general sense conventionally used, except restricted to a neighbourhod of the hyperbolic critical point, but they are flow invariant, and satisfy the properties respectively ascribed to M_s and M_u in the Stable Manifold Theorem. This is primarily because the function of which they are level sets can fail to be constant rank, and the "manifold" can intersect itself, so when we say "C^k-manifold" below, we mean essentially that it is C^k on neighbourhoods where the function defining it has the same rank.

We are also in a position to speak briefly of non-hyperbolic critical points:

Theorem 9.2 (Centre Manifold Theorem). *Let $f \in C^1(U;\mathbb{R}^d)$ and $f(\mathbf{0}) = \mathbf{0}$. Suppose* $\mathrm{D}f(\mathbf{0})$ *has k eigenvalues with negative real parts, m eigenvalues with zero real parts, and $(d-k-m)$ eigenvalues with positive real parts. There exists*

(i) *an m-dimensional C^1-CENTRE MANIFOLD $W^c(\mathbf{0})$ tangent to the centre subspace E^c of the linearized system at $\mathbf{0}$,*

(ii) *a k-dimensional C^1 stable manifold $W^s(\mathbf{0})$ tangent to the stable subspace E^s of the linearized system at $\mathbf{0}$, and*
(iii) *a $(d-k-m)$-dimensional C^1 unstable manifold $W^u(\mathbf{0})$ tangent to the unstable subspace E^u of the linearized system at $\mathbf{0}$.*

These three subsets of $\mathbb{R}^d$ are invariant under the flow ϕ_t.

What happens on the centre manifold shall remain a mystery to us as long as we are only willing to look at approximations to first order because of another topological fact, this time of the real numbers. If $\lambda = \sigma + i\tau$, and $\sigma \neq 0$, then there is always a small enough perturbation of λ by $h \in \mathbb{C}$ such that the sign of $\Re(\lambda+h)$ is the same as the sign of σ. Not being zero is an open condition. But if $\sigma = 0$, any (general) perturbation of λ will give σ a sign. Therefore, we see that what determines the behaviour on the centre manifold is determined by how the nonlinear terms perturb the system *spectrally* in a neighbourhood of a critical point. We shall find that at nonhyperbolic critical points, completely novel behaviours can arise because of nonlinearity.