

TMA4165: PROBLEM SHEET II

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1. (June 2018 examination) Let $x^0(t)$ and $x^\eta(t)$ be solutions to the Cauchy problem

$$0 = \dot{x} - x + \frac{1}{1+r^2}x^2, \quad x(0) = \frac{1}{2},$$

with $r = 0$ and $r = \eta > 0$, respectively.

Show that for each $r \in \mathbb{R}$, this equation has a unique solution on some interval of existence around $t = 0$, and that the dependence on the parameter r is continuous, in particular, that there is a bounded function $K : t \rightarrow \mathbb{R}$ independent of r such that

$$|x^0(t) - x^\eta(t)| \leq K(t)\eta^2.$$

2. Show that

$$T(x) = \frac{\pi}{2} + x - \arctan(x), \quad x \in \mathbb{R}$$

has no fixed point even though

$$|T(x) - T(y)| < |x - y|$$

for every pair of distinct $x, y \in \mathbb{R}$.

Explain if this contradicts the contraction mapping principle.

3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Show that the following *boundary value* problem for a *second-order* ODE:

$$\begin{aligned} -\frac{d^2}{dx^2}u(x) + \lambda \sin(u(x)) &= f(x), & x \in [0, 1] \\ u(0) &= 0, & u(1) = 0 \end{aligned}$$

has a unique solution for every sufficiently small λ . Write out the first three Picard iterations, beginning with $u_0 \equiv 0$.

4. (*Perko* Example 2.1) Show that the Cauchy problem

$$\frac{d}{dt}y(t) = y^2(t) - t, \quad y(0) = 1$$

exhibits finite-time blow-up, and that this maximal time of existence is less than $t = 1.13$.

5. The following equation for $f : [-a, a] \rightarrow \mathbb{R}$ arises in a model for one-dimensional gas dynamics:

$$f(x) = 1 + \frac{\pi}{2} \int_{-a}^a \frac{1}{1 + (x - y)^2} f(y) \, dy, \quad x \in [-a, a].$$

Show that the integral equation has a unique, non-negative solution for every $0 < a < \infty$. What happens in the asymptotic regime as $a \rightarrow \infty$?