22. Lecture XXII: One-Dimensional Local Bifurcations I

With this lecture we move onto the final part of the module, where we shall concern ourselves with bifurcations of systems with parameters. We have seen systems with parameters before, both fixed and vanishing. It happens that at times, the qualitative behaviour of systems change drastically and suddenly as the parameters on which the depend vary continuously. We call these phenomona BIFURCATIONS. Recall in Example 7.5, the activator-inhibitor model

$$\begin{aligned}\dot{x} &= \sigma\frac{x^2}{1+y} - x \\ \dot{y} &= \rho(x^2 - y).\end{aligned}$$

has one critical point at the origin for any σ, but only at $\sigma > 2$ does it suddenly have two further critical points at $(r_\pm, r_\pm^2)$, where

$$r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

We say that a bifurcation occurs at $\sigma = 2$ for this system.

We shall be looking systematically at simple bifurcations in the remaining lectures in this module. First we shall consider one-parameter systems, so we shall be looking at systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \tag{33}$$

where μ varies over $\mathbb{R}$, and specifically, is *not* the temporal variable. For each fixed μ, we have a C^1-autonomous, first order system of long familiarity. We may think of this as a system over $\mathbb{R}^{d+1}$ by augmenting it with the equation $\dot{\mu} = 0$.

22.1. **Dimension one/Codimension one bifurcations.** The reason that the title of this subsection seems somewhat of an oxymoron is that again, the nomenclature is a matter of perspective. In any case, the bifurcation is only happening along one direction. We shall introduce four types of bifurcations below.

To understand the codimension one perspective, first suppose the system (33) has a critical point at $(\mathbf{x}_0, \mu_0)$ at which $\mathrm{D}f(\mathbf{x}_0, \mu_0)$ (is an $d \times d$ matrix which) has a single zero eigenvalue. We know that hyperbolic critical points are quite stable already.

Then by the centre manifold theorem applied to the $(d+1)$ dimensional system, we know that there is a two dimensional manifold $W^c((\mathbf{x}_0, \mu_0)) \subseteq \mathbb{R}^{d+1}$ tangent to the centre subspace at $(\mathbf{x}_0, \mu_0)$. Restricting the system to the surface $W^c((\mathbf{x}_0, \mu_0))$, we can "foliate" $W^c((\mathbf{x}_0, \mu_0))$ be (one dimensional) curves indexed by a parameter μ close to μ_0. (That is, we can think of $W^c((\mathbf{x}_0, \mu_0))$ as being made up entirely of curves γ_μ as μ varies, which do not intersect.)

The centre manifold theorem gives us:

$$\left.\frac{\partial g}{\partial y}\right|_{(\mathbf{x}_0, \mu_0)} = 0, \tag{34}$$

where $g : \mathbb{R}^{d+1} \to \mathbb{R}$ is the "centre" component of $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$, and y is the "centre" variable that is not μ. That is, the centre equations of the $(d+1)$-dimensional system are

$$\dot{y} = g, \qquad \dot{\mu} = 0.$$

We shall now see that by increasing the "degeneracy" of this centre manifold dynamics, we naturally arrive at different types of bifurcating behaviour, in which the system changes abruptly in different ways as μ varies continuously.

1. Saddle-node bifurcation

We can stem the degeneracy on the centre manifold of the $(d+1)$-dimensional system by requiring

$$\left.\frac{\partial g}{\partial \mu}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0, \qquad \left.\frac{\partial^2 g}{\partial y^2}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0. \tag{35}$$

This is a sort of transversality condition, because we know that this means whilst the curves γ_μ with μ close enough to μ_0 all reach an optimum in the y direction on $W^c((\mathbf{x}_0,\mu))$ (this is (34)), at least they do not vanish to second order in both the y and the μ direction.

Integrating the equations (34) and (35) leads us directly to the equation for centre variable of the form:

$$\dot{y} = (\mu-\mu_0) - (y-y_0)^2 + O((\mu-\mu_0)(y-y_0), (\mu-\mu_0)^2, (y-y_0)^3),$$

where we have normalized over all constants that could be multiplied to $(\mu-\mu_0)$ or $(y-y_0)^2$. Here y_0 is the "centre" component of $\mathbf{x}_0$. The minus sign is an arbitrary convention, as we shall see.

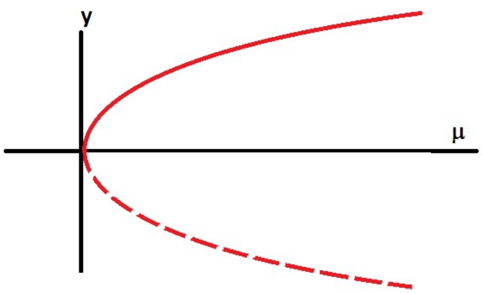
Neglecting the higher order terms, and assuming $\mathbf{x}_0 = \mathbf{0}$, $\mu_0 = 0$, there are critical points along the nullcline $y^2 = \mu$. As μ varies over $\mathbb{R}$, we see that there are no critical points for the system where $\mu < 0$. For $\mu > 0$, there are two, $y = \pm\sqrt{\mu}$. This would be reversed if the minus sign were a plus sign. At $\mu = 0$, of course, there is one single critical point, about which we had applied the centre manifold theorem to begin with.

We can ask about the stability of these two critical points where they exist. Looking again at the centre variable equation with $\mathbf{x}_0 = \mathbf{0}$ and $\mu_0 = 0$, we find that at the linearization:

$$\left.\frac{\partial}{\partial y}\right|_{\pm\sqrt{\mu}} (\mu - y^2) = 2y\big|_{\pm\sqrt{\mu}} = \mp 2\sqrt{\mu}.$$

That means that $y = -\sqrt{\mu}$ is an unstable fixed point and $y = \sqrt{\mu}$ is a stable fixed point. We do not have to worry about the non-centre variables because their characteristics are stable with respect to small perturbations around $(\mathbf{0}, 0)$.

We can record this graphically in what is known as a BIFURCATION DIAGRAM:



The red lines indicate the locations of the fixed points as μ varies away from 0. The dashed line indicates an unstable fixed point and a solid line indicates a stable fixed point. We call this type of bifurcation a SADDLE-NODE BIFURCATION. The statement that these transversality conditions guarantee a saddle-node bifurcation is known as Sotomayor's Theorem (see *Perko*, pg. 338).

2. Transcritical bifurcation

By allowing one higher order of degeneracy, we arrive at the transversality/non-degeneracy condition:

$$\left.\frac{\partial^2 g}{\partial \mu\, \partial y}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0. \tag{36}$$

Integrating the equations (34) and (36) leads us now to the following equation for the centre variable:

$$\dot{y} = \mu y - y^2 + O(\mu^2, y^3),$$

where again, we have taken $\mathbf{x}_0 = \mathbf{0}$ and $\mu_0 = 0$.

Sufficiently close to $(\mathbf{x}_0, \mu_0)$, ther are critical points at $y^2 - \mu y = 0$. That is, at $y = 0$ and at $y = \mu$. There are always two critical points as μ varies over $\mathbb{R}$, except at $\mu = 0$.

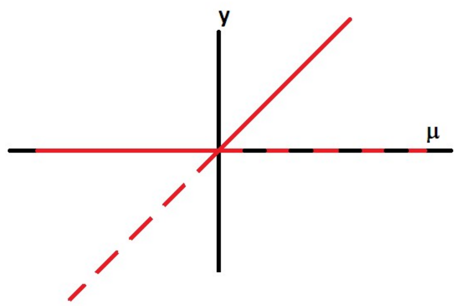
We can again analyse the stabilty of the critical points as μ varies away from 0 over $\mathbb{R}$. The linearized system is governed by

$$\frac{\partial}{\partial y}(\mu y - y^2) = \mu - 2y.$$

This derivative is positive or negative according as μ is positive or negative along $y = 0$. This means the fixed point $y = 0$ is unstable when $\mu > 0$, and stable when $\mu < 0$.

This derivative is negative or positive according as μ is positive or negative along $y = \mu$. This means that the fixed point $y = \mu$ is stable when $\mu > 0$, and unstable when $\mu < 0$.

We can again record this type of bifurcation graphically:



The red lines indicate the locations of the fixed points as μ varies away from 0. The dashed line indicates an unstable fixed point and a solid line indicates a stable fixed point. We call this type of bifurcation a TRANSCRITICAL BIFURCATION.

3. Pitchfork bifurcation

Let us continue allowing one higher order of degeneracy. We allow

$$\left.\frac{\partial^2 g}{\partial y^2}\right|_{(\mathbf{0},0)} = 0,$$

in addition to (34) but require

$$\left.\frac{\partial^3 g}{\partial y^3}\right|_{(\mathbf{0},0)} \neq 0. \tag{37}$$

These conditions then lead us as before to:

$$\dot{y} = \mu y - y^3 + O(\mu^2, y^4).$$

Sufficiently close to $(\mathbf{0}, 0)$ there are critical points at $y^3 - \mu y = 0$. That is, at $y = 0$, and at $y^2 - \mu = 0$. When $\mu < 0$, there is only one critical point. At $\mu > 0$, there are three.

The stability of $y = 0$ as μ varies away from 0 depends on

$$\left.\frac{\partial}{\partial y}\right|_{y=0} (\mu y - y^3) = \mu,$$

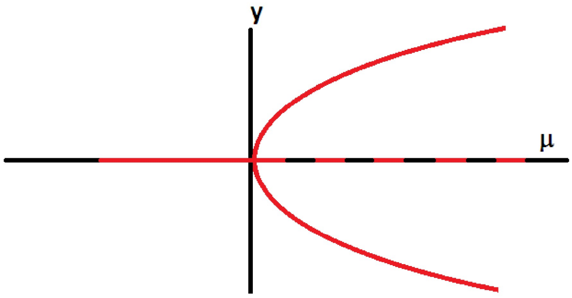
which implies stability when $\mu < 0$ and instability when $\mu > 0$.

At $y = \pm\sqrt{\mu}$, we find

$$\left.\frac{\partial}{\partial y}\right|_{y=\pm\sqrt{\mu}} (\mu y - y^3) = \mu - 3\mu = -2\mu.$$

That is, both these critical points are stable where they exist, which is only over $\mu > 0$.

This this type of bifurcation is known as the PITCHFORK BIFURCATION, and its bifurcation diagram is:



The naturality with which these bifurcations have arisen and their simplicity suggest that they arise often and in many simplified/approximate models. This is indeed the case and we shall look at some examples next time.