18.3. **Multi-scale expansion.** To secure a similar correction for Example 18.3, we shall require a generalization of the Poincaré-Lindstedt method.

In Example 18.3, we considered the Cauchy problem

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1,$$

which has the exact solution

$$x(t) = \frac{1}{\sqrt{1 - \varepsilon^2/4}} e^{-\varepsilon t/2} \sin(\sqrt{1 - \varepsilon^2/4}t).$$

Notice that there is a slowly decaying factor that happens on the order of $\varepsilon t \approx 1$ and another quicker, oscillatory behaviour on the time scale $t \approx 1$. In particular, the slow scale dynamics is not oscillatory. This suggests that perhaps we can decouple the dynamics acting on a slow time scale with the dynamics acting over a quick time scale. Therefore we introduce the variables

$$\tau_1 = t, \qquad \tau_2 = \varepsilon^\alpha t,$$

for some $\alpha > 0$.

We then postulate an ansatz of the form

$$x(t) = x(\tau_1, \tau_2) = x_0(\tau_1, \tau_2) + \varepsilon x_1(\tau_1, \tau_2) + \varepsilon^2 x_2(\tau_1, \tau_2) + \cdots.$$

Putting the ansatz into the equation, we find that

$$(\partial_{\tau_1}^2 + 2\varepsilon^\alpha \partial_{\tau_1}\partial_{\tau_2} + \varepsilon^{2\alpha}\partial_{\tau_2}^2)x(t) + \varepsilon(\partial_{\tau_1} + \varepsilon^\alpha\partial_{\tau_2})x(t) + x(t) = 0.$$

Having two times, the inital conditions no longer guarantee a unique solution. We shall use the freedom thus afforded to elimate secular terms.

Again we can sort the terms by powers of ε. To zeroth order we have

$$(\partial_{\tau_1}^2 + 1)x_0 = 0, \qquad x_0(0,0) = 1, \partial_{\tau_1}x_0(0,0) = 1.$$

The general solution is

$$x_0(\tau_1, \tau_2) = A(\tau_2)\sin(\tau_1) + B(\tau_2)\cos(\tau_1), \qquad A(0) = 1, \quad B(0) = 0. \tag{30}$$

To ε order, we have so far

$$\partial_{\tau_1}^2 x_1 + \partial_{\tau_1}x_0 + x_1 = 0.$$

The first order term imposes a secular term, and so we should like to balance this growth in τ_1 by growth in the opposite direction with τ_2. The only term available to us if $\alpha > 0$ is

$$2\varepsilon^\alpha \partial_{\tau_1}\partial_{\tau_2}x_0.$$

This forces us to choose $\alpha = 1$. And the first order equation is then modified to

$$(\partial_{\tau_1}^2 + 1)x_1 = -2\partial_{\tau_1}\partial_{\tau_2}x_0 - \partial_{\tau_1}x_0, \qquad x_1(0,0) = 0, \quad \partial_{\tau_1}x_1(0,0) = -\partial_{\tau_2}x_0(0,0).$$

This equation is not too difficult to solve when we substitute x_0 from above into it:

$$(\partial_{\tau_1}^2 + 1)x_1 = (2\dot{B} + B)\sin(\tau_1) - (2\dot{A} + A)\cos(\tau_1),$$

which yields

$$x_1(\tau_1, \tau_2) = C(\tau_2)\sin(\tau_1) + D(\tau_2)\cos(\tau_1) - \frac{1}{2}(2\dot{B} + B)\tau_1\cos(\tau_1) - \frac{1}{2}(2\dot{A} + A)\tau_1\sin(\tau_1),$$

with

$$C(0) = \dot{A}(0), \qquad D(0) = 0.$$

We can eliminate the secular terms in $x_1(\tau_1, \tau_2)$ by requiring

$$2\dot{B} + B = 0, \qquad 2\dot{A} + A = 0.$$

Along with the initial conditions in (30), we find that $B \equiv 0$ and $A(\tau_2) = \exp(-\tau_2/2) = \exp(-\varepsilon t/2)$.

This gives us the zeroth order approximation

$$x(t) \approx x_0(t) = e^{-\varepsilon t/2}\sin(t).$$

19. Lecture XIX: Perturbation Theory II

Today we shall consider the application of perturbation theory to oscillatory solutions of the van der Pol system that we encountered in Example 7.1. We shall be interested in applying to it the Poincaré-Lindstedt method in a small parameter regime. We shall also examine perturbation of systems dependent on a small parameter where the behaviour changes dramatically when $\varepsilon = 0$ is reached — systems for which there is some the limit behaviour is not well approximated by behaviours of system with small but non-zero ε. Such systems are not uncommon. Mildly viscous and entirely inviscous fluids behave very differently, for example, because a parabolic equation (which governs viscous flows),

$$\partial_t u - \varepsilon \Delta u + u \partial_x u = f(t, x, u)$$

becomes a hyperbolic equation in the $\varepsilon \to 0$ limit.

19.1. Lienard's Theorem.

Before we continue, however, it is necessary to show that the van der Pol system has periodic/oscillatory solutions. Even though we have discussed planar systems at length, and even considered the van der Pol system in the neighbourhoods of its hyperbolic fixed points, we have yet to acquire the tools that shall allow us to show that the van der Pol system has periodic solutions. We shall therefore be taking a small detour into the theory of Lienard systems.

A Lienard equation is an equation of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0.$$

Using the transformation

$$F(x) = \int_0^x f(r)\,\mathrm{d}r, \quad x = x, \quad y = \frac{\mathrm{d}x}{\mathrm{d}t} + F(x),$$

we arrive at a system of the following form,

$$\dot{x} = y - F(x)$$
$$\dot{y} = -g(x).$$

Such systems are known as Lienard systems.

Suppose that F and g satisfy the following conditions:

(i) $g, F \in C^1(\mathbb{R})$ are odd,
(ii) $xg(x) > 0$ for $x \neq 0$
(iii) $F \in C^1(\mathbb{R})$,
(iv) $F'(0) < 0$,
(v) F has a single zero at $a > 0$, and
(vi) F increases monotonically to infinity for $x \geq a$.

Theorem 19.1 (Lienard's Theorem). *A Lienard system satisfying the conditions (i) - (vi) has a unique limit cycle, and that limit cycle is stable.*

As in many of the "energy"- or "harmonic"-type methods (Lyapunov function, Hamiltonian function, function from Dulac's theorem) we have used to analyse the workings of the centre manifold, this theorem can be proven with a suitably chosen energy. Our candidate is

$$u(x, y) = \frac{1}{2}y^2 + G(x), \qquad \int_0^x g(w)\,\mathrm{d}w.$$

which is very recognizably the total energy, with a "kinetic" part, $y^2/2$, and a "potential" part G, with corresponding "force", g.

Proof. Since F and g are odd, it holds that $F(0) = g(0) = 0$. As well, $F(-a) = F(a) = 0$. Condition (ii) implies that $g(x) > 0$ for $x > 0$ and $g(x) < 0$ for $x < 0$.

Therefore any trajectory intersecting the upper or lower y-axis must intersect it perpendicularly ($\dot{y} = 0$). Any trajectory must also intesect the curve $y = F(x)$ vertically as $\dot{x} = 0$ at the intersection.

Notice that in the right half-plane, $\dot{y} < 0$, and above the curve $y = F(x)$, $\dot{x} > 0$, whilst below this curve $\dot{x} < 0$. Also notice that the transformation $(x, y) \mapsto (-x, -y)$ leaves the system unchanged.

These three things imply:

(i) that a trajectory starting at $P_0 = (0, y_0)$ on the upper y-axis will intersect $y = F(x)$ at $P_1 = (x_1, y_1)$ first, and then $P_2 = (0, y_2)$ on the lower y-axis,

(ii) this arc of the trajectory in the right-half plane is bounded by the rectangle $[0, x_1] \times [y_2, y_0]$,

(iii) and that this trajectory will continue into a closed curve if and only if $y_2 = -y_0$.

For the energy function $u(x, y)$, the closed curve condition is equivalent to $u(0, y_0) = u(0, y_2)$. Since the system is autonomous, for each point $(x_1, F(x_1))$, we can find a unique pair $(y_0(x_1), y_2(x_1))$ such that the trajectory passing through $(x_1, F(x_1))$ also passes through $(0, y_0)$ and $(0, y_2)$ with no other intersection with the y-axis in between. Define now

$$\varphi(x_1) = u(0, y_2(x_1)) - u(0, y_0(x_1)) = \int_A du,$$

where A is the arc of the trajectory described, between $(0, y_0)$ and $(0, y_2)$.

Notice that

$$du(x, y) = (y\dot{y} + g(x)\dot{x})dt = -F(x)g(x)\, dt.$$

If $x_1 \leq a$, then the entire arc is bounded between the y-axis and $y = a$. Therefore, $F < 0$ on that arc, and $g > 0$, so that $du > 0$, and $\phi(x_1) > 0$, so a trajectory passing through such a point $(x_1, F(x_1))$ cannot be not closed. Furthermore $\varphi(a) > 0$.

We shall show that φ decreases monotonically to $-\infty$ from $\varphi(a) > 0$ on $[a, -\infty)$, which would imply the existence of a value α for which $\varphi(\alpha) = 0$.

We can decompose A into three arcs, A_1 from $(0, y_0)$ to (a, v_1), A_2 from (a, v_1) to (a, v_2), and finally, A_3 from (a, v_2) to $(0, y_2)$. Along the first and last arcs,

$$\int_{A_1} du = \int_0^a -g(x)F(x)y - F(x)\, dx, \qquad \int_{A_2} du = \int_a^0 -g(x)F(x)y - F(x)\, dx = \int_0^a \frac{-g(x)F(x)}{-y + F(x)}\, dx.$$

Both of these decrease monotonically as x_1 increases beyond α because y increases in the upper arc for each x, and decreases in the lower arc for wach x. Along the arc A_2,

$$\int_{A_2} du = \int_{v_1}^{v_2} -g(x)F(x)\frac{dt}{dy}\, dy = \int_{v_2}^{v_1} -F(x)\, dy.$$

As x_1 increases, the arc moves to the right, so $F(x)$ increases monotonically to infinity, as well, $v_1 - v_2$ also increases to infinity. Therefore the integral decreases monotonically to infinity, and there are no cancellations from the other arcs.

The sign of $\varphi(x_1)$ on either side of $\varphi^{-1}(0)$ implies stability.

$\square$

Recall from Example 7.1 that the van der Pol system is

$$\dot{x} = y$$
$$\dot{y} = -\beta(x^2 - 1)y - x.$$

We can put this into Lienard form by setting $z = y + F(x)$, where $f(x) = \beta(x^2 - 1)$. Now that we know there is a limit cycle, we shall try to approximate it and analyse solutions about it.

19.2. **Revisiting the Poincaré-Lindstedt Method.** We shall write the van der Pol system as a scalar equation

$$\ddot{x}(t) + \beta(x^2 - 1)\dot{x}(t) + x(t) = 0, \qquad x(0) = a, \ \dot{x}(0) = 0.$$

Suppose $\beta > 0$ is small, and take $\tau = \omega(\beta)t$, where $\omega(\beta) = 1 + \beta\omega_1 + \beta^2\omega_2 + \cdots$. The van der Pol equation then becomes

$$\omega^2(\beta)\ddot{x}(\tau) + \beta\omega(\beta)(x^2 - 1)\dot{x}(\tau) + x(\tau) = 0.$$

Using the ansatz

$$x(\tau) = x_0(\tau) + \beta x_1(\tau) + \beta^2 x_2(\tau) + \cdots,$$

we have the initial conditions

$$x_0(0) = a, \quad 0 = \dot{x}_0(0) = x_1(0) = \dot{x}_1(0) = \cdots.$$

We can collect like powers of β and arrive at the equations:

$$\ddot{x}_0(\tau) + x_0(\tau) =$$
$$\ddot{x}_1(\tau) + x_1(\tau) = -2\omega_1\ddot{x}_1(\tau) - (x_0^2(\tau) - 1)\dot{x}_1(\tau)$$
$$\ddot{x}_2(\tau) + x_2(\tau) = -\left((\omega_1^2 + 2\omega_2)\ddot{x}_0(\tau) + 2\omega_1\ddot{x}_1(\tau)\right) - (x_0^2(\tau) - 1)\left(\dot{x}_1(\tau) + \omega_1\dot{x}_0(\tau)\right)$$
$$- 2x_0(\tau)x_1(\tau)\dot{x}_0(\tau)$$
$$\ldots.$$

As before have

$$x_0(\tau) = a\cos(\tau).$$

Substituting this solution into the second equation,

$$\ddot{x}_1(\tau) + x_1(\tau) = 2a\omega_1\cos(\tau) - a(1 - a^2/4)\sin(\tau) + (a^3/4)\sin(3\tau).$$

The degrees of freedom provided by the free choices of ω_i allow us to eliminate secular terms.

Looking more closely at the Poincaré-Lindstedt method, we shall see that if we tried to elimate resonant forces/secular terms from a non-periodic solution, we shall be unable so to do. For one of the resonant terms can be elimated by choosing $\omega_1 = 0$. But the other requires the choice $a = \pm 2$, which are the initial conditions that lands us on the limit cycle. That gives us $x_1(\tau) = \sin^3(\tau)$.

Let us consider one higher order in the expansion $\omega(\beta)$. Substituting $x_1(\tau)$ into the third equation, we arrive at

$$\ddot{x}_2(\tau) + x_2(\tau) = 2a\omega_2\cos(\tau) - (a^2\cos(\tau) - 1)3\sin^2(\tau)\cos(\tau)$$
$$- 2a^2\cos(\tau)\sin^4(\tau)$$
$$= (4\omega_2 - 11)\cos(\tau) - 31\cos^3(\tau) + 20\cos^5(\tau)$$
$$= \left(4\omega_2 - \frac{1}{4}\right)\cos(\tau) - \frac{3}{2}\cos(3\tau) + \frac{5}{4}\cos(5\tau).$$

Therefore we should take $\omega_2 = -1/16$, and have, up to order β,

$$x(t) = 2\cos(\omega t) + \beta\sin^3(\omega t) + O(\beta^2), \qquad \omega = 1 - \beta^2/16 + \cdots.$$

As expected, with β small, the limit cycle is close to a circle of radius $a = 2$.