15. Lecture XV: Limit Sets

15.1. **α and ω-limit sets.** In departing from discussion restricted to limit points, or fixed points, of a dynamical system, and dynamics in a neighbourhood of one such, we are taking a step away from the local theory to consider more global structures of systems. To this end, we shall need to begin with some language.

First, we should like to take a more abstract view of "trajectories". So far we have been using this term loosely to refer both, to a particular solution curve $\mathbf{x} : t \mapsto \phi_t(\mathbf{x}_0)$ of a system with flow ϕ, starting at an initial point $\mathbf{x}_0$, i.e., as a function of t, as well as to the set of points

$$\Gamma_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \exists t \in \mathbb{R}\ (\phi_t(\mathbf{x}_0) = \mathbf{y})\}.$$

We shall maintain this dual usage of the word "trajectory", and point out that for autonomous systems, the latter notion can be thought of as an equivalence class of solution curves modulo a time shift, as autonomous systems have trajectories that are either disjoint or coincident.

Recall that we have defined also the forward and backward orbit of a point $\mathbf{x}_0$, which we denote a little differently here:

$$\Gamma^+_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} = \phi_t(\mathbf{x}_0),\ t \geq 0\}$$
$$\Gamma^-_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} = \phi_t(\mathbf{x}_0),\ t \leq 0)\}.$$

Clearly we have

$$\Gamma_{\mathbf{x}_0} = \Gamma^+_{\mathbf{x}_0} \cup \Gamma^-_{\mathbf{x}_0}.$$

As we have seen, it is not unusual for trajectories to tend towards fixed points as $t \to \pm\infty$. We shall define two sets of limit points. The ω-limit points of $\Gamma(\mathbf{x}_0)$ are points $p \in \mathbb{R}^d$ for which there is a sequence $t_n \geq 0$ of times tending to ∞ for which

$$\lim_{n\to\infty} \phi_{t_n}(\mathbf{x}_0) = p.$$

Similarly, the α-limit points of $\Gamma(\mathbf{x}_0)$ consist of points $p \in \mathbb{R}^d$ for which there is a sequence $t_n \leq 0$ of times tending to $-\infty$ for which

$$\lim_{n\to\infty} \phi_{t_n}(\mathbf{x}_0) = p.$$

We denote these two sets of limit points by $\omega(\Gamma(\mathbf{x}_0))$ and $\alpha(\Gamma(\mathbf{x}_0))$, respectively. A time-reversal of the system interchanges ω and α. Often, the dependence on $\mathbf{x}_0$ is implicit and suppressed.

Notice that these sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are invariant with respect to ϕ_t. That is,

Lemma 15.1. *Let $p \in \mathbb{R}^d$ be an ω (resp. α)-limit point of Γ. Then $\Gamma_p \in \omega(\Gamma)$ (resp. $\in \alpha(\Gamma)$).*

This statement is pretty much obvious from the definition of an ω-limit point of Γ. We describe the main idea:

Let $\mathbf{x}_0 \in \Gamma$. Since the system is autonomous, we can take Γ to be $\Gamma_{\mathbf{x}_0}$. There is then a sequence of times $\{t_n\}$ diverging to infinity such that $\phi_{t_n}(\mathbf{x}_0) \to p$ as $n \to \infty$. Using the semigroup property of the flow of an autonomous system, if $q = \phi_T(p) \in \Gamma_p$, then $\phi_T \circ \phi_{t_n}(\mathbf{x}_0) = \phi_{t_n+T}(\mathbf{x}_0) \to q$. Therefore $q \in \omega(\Gamma)$ also.

The proof for $\alpha(\Gamma)$ is analogous.

If p is a point in $\alpha(\Gamma)$ or $\omega(\Gamma)$, then a trajectory through p is konwn as a limit orbit of Γ.

Recall that open sets on $\mathbb{R}^d$ are by definition the very sets that can be written as a union of open balls. A set is closed if it is the complement of an open set. Also by definition, $\varnothing$ is both open and closed. We say that a subset E of a closed set F is dense if the smallest closed set containing it is F itself, for example $\mathbb{Q}$ is dense in $\mathbb{R}$. We say that F is the closure of E. If E is dense in F, then for any $x \in F$ and $\varepsilon > 0$, $B_\varepsilon(x) \cap E$ is non-empty. Otherwise $F \backslash B_\varepsilon(x)$ is another closed set that contains E, strictly smaller than F. If an orbit $\phi_t(x_0)$ were dense in a cloesd set E, then it comes arbitrarily close to any point in E.

In particular, closed sets contain their limit points.

Having in mind the definition of α and ω limit sets, it should be no surprise that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are closed sets, which if bounded, are then also compact. We state a theorem to this effect:

Theorem 15.2. *Let Γ be a trajectory of a C^1-first order autonomous system. The limit sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are closed sets, and if Γ is contained in a compact subset of $\mathbb{R}^d$, then $\alpha(\Gamma)$ and $\omega(\Gamma)$ are also compact.*

This follows from the definition of $\alpha(\Gamma)$ (resp. $\omega(\Gamma)$) in pretty much the same way as the Lemma previously. We again sketch the general idea:

Suppose $\omega(\Gamma)$ were not closed, then there is a sequence $\{\mathbf{x}_n\} \subseteq \omega(\Gamma)$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^d \backslash \omega(\Gamma)$. For each n, there is a sequence $\{t_k^n\}$ of times diverging to infinity such that $\phi_{t_k^n}(\mathbf{y}_0) \to \mathbf{x}_n$, where $\mathbf{y}_0 \in \Gamma$. But then the diagonal sequence $\phi_{t_n^n}(\mathbf{y}_0)$ also converges and its limit must be in $\omega(\Gamma)$ by definition. This limit has to be $\mathbf{x}$ — a contradiction.

The point of stating the foregoing theorem is that we can define two new notions:

A closed invariant set A is an ATTRACTING SET if there is a neighbourhood U of A for which

$$p \in U \quad \implies \quad (\forall t \geq 0,\ \phi_t(p) \in U) \wedge (\phi_t(p) \to A \text{ as } t \to \infty).$$

An ATTRACTOR is an attracting set that contains a dense orbit. This essentially makes attractors smallest possible/minimal attracting sets.

By definition, every attracting set is an ω-limit set of every trajectory in a neighbourhood around it, but the converse does not hold. A simple counterexample to the converse is the fixed point of a saddle, which is an ω-limit set for three trajectories (two separatrices and the fixed point) but not any other trajectory in a small enough neighbourhood.

It is also possible for a limit set simultaneously to to be the α-limit set of some trajectories and the ω-limit set of other trajectories. It is evident that a saddle point also has this characteristic. If they are only the former, they are UNSTABLE, if only the latter, STABLE, and if both, then SEMISTABLE.

15.2. Examples.

Example 15.1. Consider the system

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 + y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2). \end{aligned}$$

First order analysis tells us that there is a fixed point at $(0,0)$, and that the linearized system exhibits an unstable focus. The Hartman-Grobman theorem for C^2-systems (or Thm. 12.2) suggests that this implies an unstable focus in a neighbourhood of $(0,0)$ for the full system.

In polar coordinates, we can see more:

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\vartheta} &= 1. \end{aligned}$$

As expected, $\dot{\vartheta} > 0$, and $\dot{r} > 0$ for $r < 1$, indicating an unstable focus near $(0,0)$, but we see that if we start at $r = 1$, we remain on the unit circle, and outside this unit circle, $\dot{r} < 0$. Therefore, the unit circle is a limit cycle that we would not have seen from just a local analysis of the system.

Example 15.2. Let us identify $\mathbb{R}^2$ with $\mathbb{C}$, and consider the discrete system constructed by the flow of

$$\dot{z} = 2\pi\alpha i z.$$

at integral times. Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

This equation can be easily integrated:

$$z(t) = e^{2\pi i \alpha t} z_0.$$

Observe that because α is irrational, $z(n)$ is dense on $|z_0|\mathbb{T}$, where $\mathbb{T} = \{\zeta : |\zeta| = 1\}$. This is a result of a theorem from diophantine analysis known as Hurwitz's Theorem:

Theorem 15.3 (Hurwitz's Theorem)**.** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then for any $Q \in \mathbb{N}$ there exists $q \leq Q$ and $p \in \mathbb{Z}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}qQ}.$$

This theorem is actually due to Dirichlet, but Hurwitz proved the best constant $1/\sqrt{5}$. Liouville extended this theorem to different orders of approximations of algebraic numbers according to their degree over $\mathbb{Q}$. Disregarding the best constant, which comes from approximating the golden ratio, it is not difficult to see why this theorem is true.

Proof. Let $[x]$ denote the largest integer smaller than of $x \in \mathbb{R}$ and set $\{x\} = x - [x]$. Given Q, since α is irrational, $\{q\alpha\}$ is never of the form n/Q. Imagine the interval $[0, 1]$ being divided up into Q subintervals of equal length. By the pigeonhole principle, as q ranges over $0, 1, 2, \ldots, Q$, at least two of the numbers $q_1\alpha$ and $q_2\alpha$ have non-integer parts that fall into the same subinterval. This means $|\{q_1\alpha\} - \{q_2\alpha\}| < 1/Q$. Putting the integers back in, there is an integer p such that

$$|(q_1 - q_2)\alpha - p| < 1/Q.$$

Setting $q = q_1 - q_2$ yields the desired approximation. □

For any $\varepsilon > 0$, we can find $Q \in \mathbb{Z}$ such that $1/Q < \varepsilon$. and for any Q, we can find n and $p \in \mathbb{Z}$ such that

$$|n\alpha - p| < 1/Q < \varepsilon.$$

So for any point with argument θ, we know that $2\pi m\alpha$ we can get within ε of θ by taking m to be an integer multiple of n above. This proves that $z(t)$ can get arbitrarily close to any point in $|z_0|C$, and hence is is dense over C.

By choosing a sequence of integral times $t_k = n_k$, we see that for any $z_0 \in \mathbb{T}$, $\phi_{t_n}(z_0)$ can converge to any point on $\mathbb{T}$, and so $\omega(\Gamma_{x_0}) = \mathbb{T}$. Notice that this is patently false for $\alpha \in \mathbb{Q}$.

This is an important example because this observation that such a flow controlled by an irrational number gives a dense, and indeed, uniformly distributed $\phi_n(z_0)$ is the fountainhead of much research in number theory, ergodic theory, and discrepancy theory.

Example 15.3. The Lorenz system was originally suggested as a model for atmospheric convection in 1963.

We briefly mention the Lorenz system, which has appeared in an exercise some weeks before:

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - \beta z,
\end{aligned}$$

where $\sigma, \rho, \beta > 0$.

Where $\alpha = \sqrt{(\rho - 1)\beta}$, we have seen that the fixed points of this system are at

$$p_1 = (0, 0, 0)^\top, \qquad p_2 = (\alpha, \alpha, \rho - 1)^\top, \qquad p_3 = (-\alpha, -\alpha, \rho - 1)^\top.$$

By inspecting the eigenvalues for the linearization at p_1, we know that the system is stable for

$$(\sigma + 1)^2 < 4\sigma(1 - \rho),$$

and unstable otherwise.

We know that the centre manifold of two dimensional analytic systems can be characterized more or less completely, even if behaviours can differ from linear systems substantially. For appropriate values of σ, ρ and β, the Lorenz system has a curiously complicated attractor unrivalled in complication by behaviours that two dimensions can support. This is the case even though one of the equations of the system is linear, and the remaining are also analytic.

The attractor A of this system is made of an infinite number of banched surfaces each of which intersect. Trajectories, being of an autonomous system, do not intersect as they move along A, but there are nevertheless periodic trajectories of arbitrarily large period, uncountably many non-periodic trajectories, and also trajectories that are dense in A. We refer to these attractors as "strange attractors".

Next we turn to a fuller discussion of limit cycles but confine ourselves to the plane.

15.3. **Limit Cycles.** CYCLES, or PERIODIC ORBITS are closed-curve solutions that are not equilibria, a definition exactly in line with our previous usage of the term. Since our systems are autonomous, if the trajectory Γ is a cycle, there is a minimal T independent of $\mathbf{x}_0 \in \Gamma$ for which $\phi_{t+T}(\mathbf{x}_0) = \phi_t(\mathbf{x}_0)$. This minimum T we call the PERIOD of the Γ. For centres of linear systems, the period is constant over a family of periodic orbits. This is not so in general. We shall be spending the next few lectures looking at the behaviour of periodic solutions and limit cycles, especially in two dimensions.

A cycle can be itself unstable, stable, and asmptotically stable. To discuss these notions analogously to the way we did for fixed points, we digress briefly to mention that the distance from a point $\mathbf{x}$ to a set E is defined as

$$d(\mathbf{x}, E) := \inf_{\mathbf{y}\in E} |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^d}.$$

Then we say that a cycle Γ is STABLE if for every $\varepsilon > 0$, there is a neighbourhood U of Γ (an open set U containing Γ) for which $\mathbf{x} \in U$ implies

$$d(\phi_t(\mathbf{x}), \Gamma) < \varepsilon.$$

The cycle Γ is UNSTABLE if it is not stable. And analogous to asymptotic stability previously defined, Γ is ASYMPTOTICALLY STABLE if it contains the ω-limit set of every trajectory within a certain neighbourhood U of itself. That is, Γ is stable, and there exists a neighbourhood U of Γ such that $\mathbf{x} \in U$ implies

$$\lim_{t\to\infty} d(\phi_t(\mathbf{x}), \Gamma) = 0.$$

Closed curves that are equilibria of sorts (not an equilibrium in the sense that they sit at the intersection of nullclines!) constitute an important class of limit sets known as LIMIT CYCLES, which we have encountered in the specific context of centre-foci. These are cycles that are the α or ω set of some trajectories. And if the entire cycle is a limit set, then there is also the notion of semistability that can be defined. If there exists a neighbourhood U of Γ for which Γ is the ω-limit set (resp. α-limit set) for every trajectory in/passing through U, then Γ is an ω-LIMIT CYCLE, or a STABLE LIMIT CYCLE (resp. α-LIMIT CYCLE, or a UNSTABLE LIMIT CYCLE). If Γ is an α-limit set for one trajectory and an ω-limit set for another trajectory, then we say that it is a SEMISTABLE LIMIT CYCLE.

Cycles, like points, have stable and unstable manifolds. For a cycle Γ, and U a neighbourhood thereof, we define the local stable and unstable manifolds as

$$M_s(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to\infty}{\to} 0,\ \forall t \geq 0,\ \phi_t(\mathbf{x}) \in U\},$$

and

$$M_u(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to-\infty}{\to} 0,\ \forall t \leq 0,\ \phi_t(\mathbf{x}) \in U\},$$

respectively. The global stable and unstable manifolds of a cycle are then

$$W^s(\Gamma) = \bigcup_{t\leq 0} \phi_t(M_s(\Gamma))$$

$$W^u(\Gamma) = \bigcup_{t\geq 0} \phi_t(M_u(\Gamma)).$$

This definition makes the global stable and unstable manifolds invariant under ϕ_t. We shall see that in many ways, we can treat a periodic orbit as a critical point, even though there are important ways that we cannot so do.