10. Lecture X: The Method of Lyapunov

10.1. **Lyapunov functions.** Having looked in some detail at hyperbolic critical points, and seeing that first order methods suffice to determine a wealth of information concerning the system in a neighbourhood of any such point, we turn now to a method that will shed some light on behviour near nonhyperbolic critical points.

First we shall refine our notions of stability. Let us call a fixed point $\mathbf{x}_0$ of the flow ϕ_t of an autonomous system STABLE if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $t \geq 0$,

$$\mathbf{y} \in B_\delta(\mathbf{x}_0) \quad \Longrightarrow \quad \phi_t(\mathbf{y}) \in B_\varepsilon(\mathbf{x}_0)\Big(= B_\varepsilon(\phi_t(\mathbf{x}_0))\Big),$$

otherwise the fixed point is UNSTABLE. This is a notion with which we are already familiar. Let us further say that $\mathbf{x}_0$ is ASYMPTOTICALLY STABLE if it is stable and if there exists a $\delta > 0$ such that in fact as $t \to \infty$, we have

$$\mathbf{y} \in B_\delta(\mathbf{x}_0) \quad \Longrightarrow \quad \lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{x}_0.$$

We know from any of the three preceding theorems on stable manifolds, centre manifolds, or the theorem of Hartman-Grobman that a hyperbolic critical point is either unstable, or otherwise asymptotically stable. However, fixed points for linear systems that exhibit centres, for example, are stable without being asymptotically stable.

The method of Lyapunov shall be able to help us to make a distinction between stable nonhyperbolic critical points that are asymptotically stable, and those which are not so.

Theorem 10.1. *Let U be a neighbourhood of a fixed point $\mathbf{x}_0$ containing only one fixed point. Let $f \in C^1(U)$ determine an autonomous system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t)),$$

and suppose that a function $V \in C^1(U)$ exists for which $V(\mathbf{y}) > V(\mathbf{x}_0)$ for $\mathbf{y} \in U$. If, furthermore,

(i) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) \leq 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is stable;*
(ii) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) < 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is asymptotically stable; and*
(iii) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) > 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is unstable.*

Since the system is autonomous,

$$\frac{\mathrm{d}V \circ \mathbf{x}}{\mathrm{d}t} = \mathrm{D}V|_{\mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{D}V|_{\mathbf{x}} \cdot f(\mathbf{x}),$$

and so the conditions in the theorem statement are a condition on the increase or decreas of $V \circ \mathbf{x}(t)$ along a trajectory $\mathbf{x}(t) = \phi_t(\mathbf{y}_0)$. The applicability of the theorem depends on finding such a function V and a neighbourhood U for which V and its derivatives has the requisite signs over the entire region U. A function V satisfying (i) or (ii) is known as a LYAPUNOV FUNCTION of the system.

Given our discussion following the statement of the Centre Manifold Theorem (Thm.9.2) last time, we can see that if we can find a Lyapunov function around a non-hyperbolic fixed point of a nonlinear system, we shall be able differentiate between centres and foci.

Before we prove the theorem we require a result on continuous functions.

Lemma 10.2. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $E \subseteq \mathbb{R}$ and $S \subseteq \mathbb{R}^d$ be a closed sets, furthermore let S be bounded (i.e., contained within $B_R(\mathbf{0})$ for a large enough finite R). Then*

(i) *$V^{-1}(E)$ is closed in $\mathbb{R}^d$, and*
(ii) *V attains its supremum in S (i.e., $\exists \mathbf{w} \in S$ such that $V(\mathbf{w}) = \sup_{\mathbf{y}\in S} V(\mathbf{y})$).*

As in Lecture 8, by open sets we mean sets that can be made up of an arbitrary union of balls, and by closed sets we mean any set that can be written as a complement of an open set.

Proof. [not examinable] If $\{x_n\}$ is a sequence contained in E that converges to x, it hold that $x \in E$ because otherwise x is contained in some open ball B_δ outside of E, and this means $|x_n - x| > \delta$ for every n, no matter how large. This property therefore characterizes closed sets.

Now suppose $V^{-1}(E)$ is not closed. Then there is a sequence $\mathbf{y}_n \to \mathbf{y}$ for which $\mathbf{y}_n \in V^{-1}(E)$ for every n but $\mathbf{y} \notin V^{-1}(E)$. Then $V(\mathbf{y}) \notin E$. Continuity transfers convergence, but $V(\mathbf{y})$ is not in E so $V(\mathbf{y}_n)$ cannot converge to E. This is a contradiction on the continuity of V.

Let $m = \sup_{\mathbf{y}\in S} V(\mathbf{y})$. Consider the sequence $\{m - 1/n\}_{n=1}^{\infty}$. The inverse image of this sequence is a cloesd set $V^{-1}(\{m - 1/n\}_{n=1}^{\infty})$. Intersected with S, one obtains a bounded closed set. From each non-empty $V^{-1}(m - 1/n) \cap S$ choose a point a_n. Since S is bounded, it can be halved by a codimension one subsapce so that infinitely many $\{a_n\}$ are in one half. This half can be halved again, ad infinitum, so that the sequence $\{a_n\}$ is seen necessarily to have a convergent subsequence. Since S is close, this convergent subsequence converges to a point $b \in S$. For any $N > 0$, there exists an M such that neighbourhood of $V(b)$ includes $m - 1/M$. This allows us to conclude by continuity that $V(b) = m$. □

Proof of Thm. 10.1. Without loss of generality, we can assume $V(\mathbf{x}_0) = 0$, and $V(\mathbf{y}) > 0$ for $\mathbf{y} \in U$ by adding a constant to V.

Part (i):

Since U is open, there is a sufficiently small ε such that U contains the closed ball $\bar{B}_\varepsilon(\mathbf{x}_0)$. Since $V(\mathbf{y}) > 0$ on $U\backslash\{\mathbf{x}_0\}$, there is a positive minimum $m_\varepsilon > 0$ to the set

$$\{V(\mathbf{y}) : |\mathbf{y} - \mathbf{x}_0| = \varepsilon\} \subseteq \mathbb{R}.$$

Since $V(\mathbf{x}_0) = 0$, by the continuity of V, for a sufficiently small $\delta < \varepsilon$, every $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ satisfies

$$V(\mathbf{y}_0) < m_\varepsilon.$$

By the non-increasing property of $V(\phi_t(\mathbf{y}_0))$, it holds that

$$V(\phi_t(\mathbf{y}_0)) \leq V(\mathbf{y}_0) < m_\varepsilon.$$

This ensures that $\phi_t(y_0)$ is never in the set $\{\mathbf{y} : |\mathbf{y} - \mathbf{x}_0| = \varepsilon\}$. From which we can conclude that $\phi_t(\mathbf{y}_0) \in B_\varepsilon(\mathbf{x}_0)$ for all time, and $\mathbf{x}_0$ is therefore a stable fixed point.

Part (ii):

From the above we know that if $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ for some sufficiently small δ, $\phi_t(\mathbf{y}_0)$ remains in $B_\varepsilon(\mathbf{x}_0) \subseteq U$.

If $V(\phi_t(\mathbf{y}_0))$ is strictly decreasing along trajectories, and it is lower bounded by $V(\mathbf{x}_0) = 0$, it holds that $V(\phi_t(\mathbf{y}_0))$ must tend to a limit as $t \to \infty$. Suppose $V(\phi_t(\mathbf{y}_0)) \to m > 0$. Then for any $\eta > 0$, we can find a sufficiently large T such that if $\tau > T$,

$$\left| \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=\tau} V(\phi_t(\mathbf{y}_0)) \right| < \eta.$$

We now derive a contradiction, the idea being that, supposing $V(\phi_t(\mathbf{y}_0))$ has to slow down to zero as it nears the set $\{\mathbf{y} : V($
$y) = m\}$, yet we can pick a point arbitrarily close to this set and show that the starting speed must be some magnitude uniformly bounded away from 0.

Observe that by continuity,

$$\mathcal{A} = \{\mathbf{y} : V(\mathbf{y}) = m\} \cap \bar{B}_\varepsilon(\mathbf{x}_0)$$

is closed and bounded, so that on this set, the continuous function (recall that $V \in C^1$)

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} V(\phi_t(\mathbf{y}))$$

attains a minimum, say $\vartheta > 0$. Therefore, we can always find a δ' neighbourhood of this set on which

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} V \circ \phi_t > \vartheta'.$$

But we can choose $\eta < \vartheta'$ so that for large enough τ, as $V(\phi_\tau(\mathbf{y}_0))$ approaches $\mathcal{A}$,

$$\left|\eta > \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} V(\phi_t(\mathbf{y}_0))\right| > \vartheta' > \eta,$$

a clear contradiction.

Part (iii):

Essentially reversing the signs/result of (ii).

□

10.2. **Examples.** As mentioned, applying the theorem turns on finding a Lyapunov function. There is no general way of doing so. We now look at two instructive examples.

Example 10.1. We consider the system

$$\begin{aligned} \dot{x} &= -2y + yz - x^3 \\ \dot{y} &= x - xz - y^3 \\ \dot{z} &= xy - z^3 \end{aligned} \quad .$$

There is a fixed point at the origin. We can find $\mathrm{D}f(\mathbf{0})$:

$$\mathrm{D}f(\mathbf{0}) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_\pm = \pm 2i$. This is a non-hyperbolic fixed point for which the entire space is locally part of the centre manifold.

We consider the function $V(x, y, z) = x^2 + 2y^2 + z^2$. Setting $(x(t), y(t), z(t)) = \phi_t(\boldsymbol{\beta})$ for $\boldsymbol{\beta}$ in a sufficiently small neighbourhood of $\mathbf{0}$, the derivative can be computed as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(\phi_t(\boldsymbol{\beta})) &= \mathrm{D}V(x, y, z) \cdot f(x, y, z) \\ &= 2x(-2y + yz - x^3) + 4y(x - xz - y^3) + 2z(xy - z^3) \\ &= -2x^4 - 4y^4 - 2z^4 \\ &< 0 \end{aligned}$$

for $(x, y, z) \neq \mathbf{0}$.

Therefore the origin is asymptotically stable, even though it is not a sink, and we see that asymptotic stability does not necessarily imply the existence of a sink (stable focus or node) at non-hyperbolic critical points of nonlinear systems.