## References

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## Contents

## 1. Lecture I: Linear Systems on $\mathbb{R}^2$ I

1.1. **Phase Spaces and Phase Flows.** Let $X$ be a locally compact Hausdorff space. Let $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times X \to X$ be a continuous map with the cocycle property:

$$\begin{aligned} \phi(t,t) &= \mathrm{Id}_X, \\ \phi(t,u) \circ \phi(u,s) &= \phi(t,s) \qquad u,t,s \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{1}$$

Here $\phi$ takes in a terminal time and an initial time in its first and second arguments respectively, and acts upon an element $\mathbf{x} \in X$ by translating it to another point $\mathbf{y} \in X$. The pair $(X, \phi)$ is a (continuous-time) dynamical system.

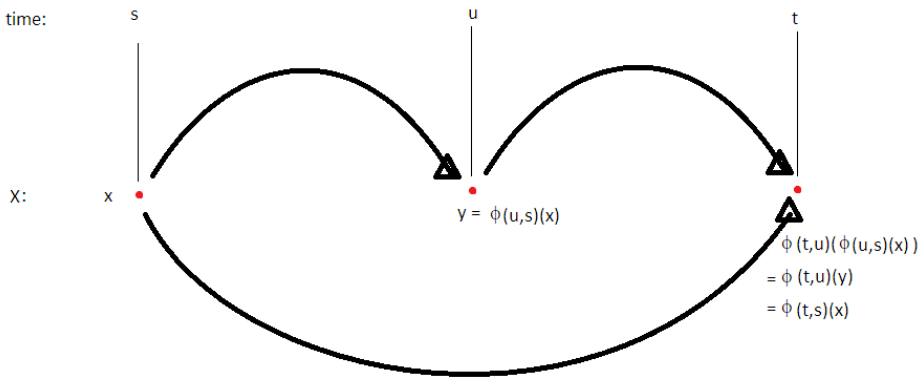



Figure 1. The cocycle property is a consistency criterion.

Taking $X = \mathbb{R}^d$, we see that the system of first-order differential equations with (locally Lipschitz) $f : [0,T] \times X \to \mathbb{R}^d$,

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t)), \tag{2}$$

can be construed as a dynamical system (if it has unique continuous solutions) by considering the pair $(\mathbb{R}^d, S)$ where $S$ is the solution map $S : (t, s, \mathbf{x}(s)) \mapsto \mathbf{x}(t)$. (There is good reason not to think of the target space of $f$ as $X$ — it is in fact the tangent space of $X$, which only happens to be identifiable with $X$ if $X$ is $\mathbb{R}^d$, but this is not true of more general spaces.)

Using the definition of $\phi$, heuristically, we can also write (2) as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(t,0,\mathbf{x}(0))} \phi = f(t, \phi|_{(t,0,\mathbf{x}(0))}). \tag{3}$$

It is sometimes convenient to define $\phi$ satisfying this equation to be the flow, or phase flow, of the function $f$. We think of (2) as giving a description of a system for an aggregate of particles from a fixed reference frame, and we think of (3) as a description of the system given by following a specific particle starting at position $\mathbf{x}(0)$. The first is known as the Eulerian description of a system and the second as the Lagrangian description. In Lectures VIII and IX, we shall be putting this framework into more geometric terms. We also call $\mathbf{x}(t)$ the integral curve, or simply an integal, of $f$.

This terminology becomes reasonable when we consider the phase space, which is simply $X$, when it houses the curves $\Gamma_+(\mathbf{x}) = \{\phi(t, 0, \mathbf{x}(0)) : t \in \mathbb{R}_{\geq 0}\}$. These curves are the forward orbits of the dynamical system, and the plots are known as phase portraits.

The object of this module is then twofold:

(i) to study qualitative changes of phase portraits as they depend on $f$, and

(ii) to study the asymptotic behaviour of the solution/integral curve $\mathbf{x}(t)$.

To these ends we shall be looking at solution curves to differential equations that blow up to infinity asymptotically or in finite time, and spend extra effort exploring the complicated behaviours that can occur when solution curves do neither of the foregoing. Where $d = 2$, it is also practicable to draw phase portraits for an important class of systems, and we shall be applying ourselves to that activity. We shall first be considering linear systems, i.e., where $f$ is linear in $\mathbf{x}$, and then consider more general systems by looking at linearization of $f$, i.e., using the approximation $f(\mathbf{x}) \approx f(\mathbf{y}) + \nabla f(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$, near fixed points $\mathbf{y}$ of particular interest. Some behaviours of nonlinear systems are so complicated that it is impossible to reduce them to their linearizations. That shall also be of interest to us, and we shall look at some techniques to characterize these behaviours.

If we have time we shall also be looking at more general dynamical systems not necessaily described by differential equations. Of those that are, we shall restrict our attention to *ordinary* differential equations.

1.2. **Structure of this module.** In this module we shall be looking at, in order,

(i) linear systems,
(ii) local theory of nonlinear systems, that is, behaviour of systems around certain "critical" values (via linearization),
(iii) global theory of nonlinear systems, and
(iv) bifurcation theory for systems governed by flows.



1.3. **Matrix exponentiation.** The system of equations (2) is quite general in the sense that for a differential equation of any finite order $d$, it is possible to recast it as a system of $d$ first-order equations by the change-of-variables $x^{i+1} = \mathrm{d}^i x/\mathrm{d}t^i$ for $i = 0, \dots d-1$, and $\mathbf{x} = (x^1, x^2, \dots x^d)$ (superscripts are indices, not powers).

Our attitude towards linear systems on $X = \mathbb{R}^d$ will be to seek to derive solutions explicitly rather than concern ourselves with the general and abstract question of existence and uniqueness. By linearity we mean

$$f(t, \mathbf{x}(t)) = \mathbf{A}(t)\mathbf{x}(t),$$

where $\mathbf{A}(t)$ is a $d \times d$ matrix, and $\mathbf{x}(t) = (x^1(t), \dots, x^d(t))^\top$.

First we impose an additional condition of autonomy. We say that a system is AUTONOMOUS if $f(t, \mathbf{x}(t)) = \tilde{f}(\mathbf{x}(t))$, or $\mathbf{A}(t) = \mathbf{A}$ in the linear case, i.e., there is no dependence on $t$ except through the solution $\mathbf{x}(t)$ — there is no direct dependence of the system on $t$. This means that the rules governing how a particle moves do not themselves change over time. This additional requirement of autonomy (which can be imposed without linearity) reduces the cocycle property (1) on the flow $\phi$ to a simpler for which $\phi : \mathbb{R}_{\geq 0} \times X \to X$, in which the flow only records the *duration* and not the terminal and initial times, and for which

$$\begin{aligned} \phi_0 &= \mathrm{Id}_X, \\ \phi_t \circ \phi_s &= \phi_{t+s} \qquad t, s \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{4}$$

It is conventional to write the temporal argument in the subscript.

Autonomy means further that $\mathbf{A}$ is *time-independent*.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t), \tag{5}$$

then, the $d = 1$ case suggests we can write down a solution

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0),$$

if we can properly interpret $\exp(\mathbf{A}t)$. If this were the case, the map $\exp(\mathbf{A}t) : \mathbb{R}^d \to \mathbb{R}^d$ can then be identified with the flow $\phi_t$ of the autonomous system.

We shall now devote some time to showing what $\exp(\mathbf{A}t)$ must mean and why this intuitive form of the solution is helpful in determining asymptotic behaviour.

Using the Taylor expansion for the exponential, we shall tentatively define

$$\exp(\mathbf{A}t) := \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n.$$

We shall require a few things:

(i) The series converges in appropriate topologies/norms.
(ii) The series satisfies number of properties expected of exponentials under appropriate conditions.
(iii) The series applied to $\mathbf{x}(0)$ is a solution to the initial value problem $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{A}\mathbf{x}$.

We shall also study the properties of exponentiations of diagonalizable matrices and, more generally, of two similar matrices.

We shall say that a sequence of $d \times d$ matrices $\{\mathbf{M}_k\}_{k=1}^{\infty}$ converges to $\mathbf{M}$ if

$$\|\mathbf{M}_k - \mathbf{M}\|_{\ell^2\to\ell^2} := \sup_{|\mathbf{x}|\leq 1} \|(\mathbf{M}_k - \mathbf{M})\mathbf{x}\|_{\ell^2} \left( = \sup_{\mathbf{x}\in\mathbb{R}^d\setminus\{0\}} \frac{\|(\mathbf{M}_k - \mathbf{M})\mathbf{x}\|_{\ell^2}}{\|\mathbf{x}\|_{\ell^2}} \right) \to 0.$$

We think of real $d \times d$ matrices as maps from $\ell^2(\mathbb{R}^d)$ to itself, and this norm is known as the OPERATOR NORM of linear operators $\ell^2(\mathbb{R}^d) \to \ell^2(\mathbb{R}^d)$ (here abbreviated to $\ell^2 \to \ell^2$ as the underlying field is static). (You can check that this is a norm.) The topology defined by convergence under the operator norm is known as the STRONG OPERATOR TOPOLOGY.

First, it can be readily verified that this norm is equivalent to the uniform norm:

$$\|\mathbf{M}_k - \mathbf{M}\|_{\ell^2\to\ell^\infty} \leq \|\mathbf{M}_k - \mathbf{M}\|_{\ell^2\to\ell^2} \leq d\|\mathbf{M}_k - \mathbf{M}\|_{\ell^2\to\ell^\infty}.$$

The space of $d \times d$ real matrices is complete under the uniform norm by the completeness of real numbers, and so it is complete under the $\ell^2 \to \ell^2$ norm. (Equivalent norms induce equivalent topologies — because of this equivalence we often drop the subscript on the norm.)

Secondly, by normalising a vector $\mathbf{x}$, it can be seen that

$$\|\mathbf{M}\mathbf{x}\|_{\ell^2} \leq \|\mathbf{M}\|_{\ell^2\to\ell^2}\|\mathbf{x}\|_{\ell^2},$$

and therefore, for two $d \times d$ matrices $\mathbf{M}$ and $\mathbf{N}$,

$$\|\mathbf{M}\mathbf{N}\|_{\ell^2\to\ell^2} \leq \|\mathbf{M}\|_{\ell^2\to\ell^2}\|\mathbf{N}\|_{\ell^2\to\ell^2}.$$

We say that the linear maps $\ell^2(\mathbb{R}^d) \to \ell^2(\mathbb{R}^d)$ form an ALGEBRA under the operator norm.

Now for any finite $k$, set

$$\mathbf{M}_k = \sum_{n=0}^{k} \frac{t^n}{n!}\mathbf{A}^n.$$

We then have

$$\|\mathbf{M}_k\|_{\ell^2\to\ell^2} \leq \sum_{n=0}^{k} \frac{|t^n|}{n!}\|\mathbf{A}\|^n_{\ell^2\to\ell^2},$$

by the triangle inequality and the algebra property.

Next, for any $\varepsilon > 0$, we can find $N$ such that

$$\|\mathbf{M}_M - \mathbf{M}_{M+N}\|_{\ell^2\to\ell^2} = \left\| \sum_{n=M+1}^{M+N} \frac{t^n}{n!}\mathbf{A}^n \right\|_{\ell^2\to\ell^2} \leq \sum_{n=M+1}^{M+N} \frac{t^n}{n!}\|\mathbf{A}\|^n_{\ell^2\to\ell^2}.$$

Therefore $\{M_k\}$ are a Cauchy sequence in the norm and by completeness the sequence, and hence the sum, converges (in fact, converges absolutely).

Finally, to explain some parts of the next lemma, let us mention that matrix-valued functions $\varphi : \mathbb{R} \to \mathbb{R}^{d\times d}$ are DIFFERENTIABLE at $t_0$ if there exists a $d \times d$ matrix $\mathbf{N}$ such that

$$\lim_{t\to t_0} \left\| \mathbf{N} - \frac{1}{t-t_0}(\varphi(t) - \varphi(t_0)) \right\|_{\ell^2\to\ell^2} = 0.$$

We write $\varphi'(t)$ for the matrix $\mathbf{N}$. Note that we could have done this element-wise as we have shown that the $\ell^2 \to \ell^2$ norm and the uniform norm are equivalent norms in finite dimensions.

Now that we have a convergent series defining $\exp(\mathbf{A}t)$ we can easily show that

**Lemma 1.1.** (i) *Let $\varphi(t)$ and $\psi(t)$ be differentiable functions $\mathbb{R} \to \mathbb{R}^{d\times d}$. Then the Leibnitz rule holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)\psi(t)\big) = \varphi'(t)\psi(t) + \varphi(t)\psi'(t).$$

*(Again, note that $\phi'$ and $\psi'$, whilst being $d \times d$ real matrices, reside in fact in the tangent space of the space of $d \times d$ matrices.)*

(ii) *Let $\mathbf{0}_d$ denote the zero matrix. Then*

$$\exp(\mathbf{0}_d) = \mathbf{I}_d.$$

(iii) *For any $d \times d$ matrix $\mathbf{A}$,*

$$\Big(\exp(\mathbf{A})\Big)^{-1} = \exp(-\mathbf{A}).$$

(vi) *If matrices $\mathbf{A}$ and $\mathbf{B}$ commute, then*

$$\exp(\mathbf{A}t)\exp(\mathbf{B}t) = \exp((\mathbf{A}+\mathbf{B})t).$$

(v) *Let $\mathbf{P}$ be a nonsingular $d \times d$ matrix, and set $\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. (That is, $\mathbf{\Lambda} \sim \mathbf{A}$ — i.e., the matrices are similar.) Then*

$$\exp(\mathbf{\Lambda}) = \mathbf{P}^{-1}\exp(\mathbf{A})\mathbf{P}.$$



*Proof.* (i) This holds by direct calculation element-wise.

(ii) The norm $\|\mathbf{0}_d\|_{\ell^2\to\ell^2}$ can be explicitly found to be 0. The statement follows by the series expansion.

(iii) Take $\psi(t) = \exp(-\mathbf{A}t)\exp(\mathbf{A}t)$ and find that its derivative is 0 via the Leibnitz rule and use (ii) with $t = 0$.

(iv) Take $\psi(t) = \exp(-(\mathbf{A}+\mathbf{B})t)\exp(\mathbf{A}t)\exp(\mathbf{B}t)$ and find that its derivative is 0 via the Leibnitz rule and use (ii). Naturally within the parenthesis $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. Therefore the result cannot be hoped to be true in general.

(v) Using the series expansion we see that the result is true for a partial sum. Then we take a limit.

□

**Theorem 1.2** (Fundamental theorem for Linear Systems)**.** *For every $\mathbf{b} \in \mathbb{R}^d$, the initial value problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{b}$$

*has a unique solution*

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{b}.$$

*Proof.* It can be readily verified that $\mathbf{x}$ as constructed is a solution.

Now suppose $\mathbf{x}$ is a solution. Then set $\mathbf{y}(t) = \exp(-\mathbf{A}t)\mathbf{x}(t)$.

Differentiating $\mathbf{y}(t)$ from first principles we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = -\exp(-\mathbf{A}t)\mathbf{A}\mathbf{x}(t) + \exp(-\mathbf{A}t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t).$$

We can use the equation to expand the final term to find that the derivative is the zero vector. This implies that

$$\mathbf{y}(t) = \mathbf{y}(0) = \mathbf{x}(0),$$

and by construction $\mathbf{x}(t)$ must take the form required by the theorem statement. □

## 2. Lecture II: Linear Systems on $\mathbb{R}^2$ II

2.1. **Autonomous linear systems on $\mathbb{R}^2$.** Part (v) of Lemma 1.1 allows us to decouple the equation (5) when $\mathbf{A}$ is diagonalizable. In fact an easier way to see it is as follows. Suppose that for a non-singular matrix $\mathbf{P}$,

$$\left(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right)_i^j = \lambda^j \delta_i^j,$$

where $\delta_i^j$ is the Kronecker Delta that is zero everywhere only except at $i = j$, when it is unity, then the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{P}\mathbf{x}\Big)^j = \lambda^j \delta_i^j \Big(\mathbf{P}\mathbf{x}\Big)^i.$$

These are $d$ decoupled one-dimensional equations for the new variable $\mathbf{y} = \mathbf{P}\mathbf{x} = (y^1, \ldots, y^d)^\top$:

$$\frac{\mathrm{d}y^j}{\mathrm{d}t} = \lambda^j y^j(t),$$

with solutions

$$y^j(t) = e^{\lambda^j t} y^j(0).$$

The matrix with entries $e^{\lambda^j t}\delta_i^j$ is sometimes written as $\mathrm{diag}(\exp(\lambda^j t))$. From the series representation it is clear that this is the same as $\exp(\mathrm{diag}(\lambda^j t))$.

In other words,

$$\mathbf{x}(t) = \mathbf{P}\exp(\mathbf{\Lambda} t)\mathbf{P}^{-1}\mathbf{x}(0) = \mathbf{P}\mathrm{diag}\big(\exp(\lambda^j t)\big)\mathbf{P}^{-1}\mathbf{x}(0). \tag{6}$$

We use the Einstein summation convention and sum up implicitly over repeated indices when one is an upper index and another is a lower index. Column vectors have upper indices and matrices have one upper and one lower index — we shall see why this is a good convention later. Context can usually inform us whether an upper index is just such or whether it is an exponent.

This derivation avoids the difficulties of having to re-derive some results when $\lambda^j$ are complex because we solve the equation after decoupling, and do not refer to the exponentiation of a complex matrix $\exp(\mathbf{\Lambda})$, where of course, $(\mathbf{\Lambda})_i^j = \lambda^j \delta_i^j$, and the associated convergence issues in $\ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$. Naturally, it is also easy to show that all the previous derivations on convergence carry over.

Decoupling as shown does not happen when $\mathbf{A}$ is not diagonalizable, of course. In that instance it becomes helpful to resort to a Jordan normal form representation of $\mathbf{A}$, and we postpone that discussion in general to Lecture 3. For the remainder of this and the next lecture, we shall focus on the case $d = 2$.

When $d = 2$, the matrix $\mathbf{A}$ in (5) is

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

We can find its eigenvalues $\lambda$:

$$\begin{aligned} 0 &= \det\Big(\lambda \mathbf{I}_2 - \mathbf{A}\Big) \\ &= \det\begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} \\ &= \lambda^2 - (a+d)\lambda + (ad - bc). \end{aligned}$$

Therefore

$$\lambda_\pm = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{1}{2}\big((a+d) \pm \sqrt{(a-d)^2 + 4bc}\big).$$

This reduces to three cases:

(A) Two distinct real roots — $(a-d)^2 > -4bc$;
(B) Root with multiplicity — $(a-d)^2 = -4bc$ — this can only happen if $bc \le 0$;

(C) Conjugate roots – $(a-d)^2 < -4bc$.

*The case (A).*

Consider now the foregoing discussion on decoupling. If there are two distinct real roots $\lambda_\pm$, there are two associated eigenvectors $\mathbf{v}_\pm$. Accordingly, we can write down the solution as

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-, \tag{7}$$

where the constants are determined by the initial condition

$$\mathbf{x}(0) = C_1\mathbf{v}_+ + C_2\mathbf{v}_-.$$

Of course all this could have been derived using, e.g., the Laplace transform, for the second-order equation by reversing the procedure for reducing equations to linear first-order systems described at the beginning of §1.3.

*The case (B).*

This case splits further into two: the first, where for the root $\lambda$ with multiplicity 2, there are associated eigenvectors that span $\mathbb{R}^2$ ("geometric multiplicity = algebraic multiplity), can be handled as in (A); and the second, where $\lambda$ does not have two linearly independent eigenvectors ("geometric multiplicity < algebraic multiplicity"), and the matrix is not in fact diagonalizable, we have to find an extra, generalized eigenvector.

From introductory linear algebra, we know that a Jordan chain gives us a second linearly independent vector so that if $\mathbf{v}_1$ is an eigenvector, then a generalized eigenvector $\mathbf{v}_2$ satisfies

$$(\mathbf{A} - \lambda\mathbf{I}_2)\mathbf{v}_2 = \mathbf{v}_1.$$

As $(\mathbf{A} - \lambda\mathbf{I}_2)$ is singular, $\mathbf{v}_2$ is not unique.

Writing

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2,$$

we can equate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f'(t)\mathbf{v}_1 + g'(t)\mathbf{v}_2$$

with

$$\mathbf{A}\mathbf{x} = f(t)\lambda\mathbf{v}_1 + g(t)\mathbf{v}_1 + g(t)\lambda\mathbf{v}_2$$

to arrive at the equations

$$\begin{aligned} f'(t) &= \lambda f(t) + g(t) \\ g'(t) &= \lambda g(t). \end{aligned}$$

This gives us the general solutions:

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2 = (C_1 + C_2 t)e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2. \tag{8}$$

Again, the constants $C_1$ and $C_2$ are determined by the initial condition via

$$\mathbf{x}(0) = C_1\mathbf{v}_1 + C_2\mathbf{v}_2.$$

*The case (C).*

The case (C) is exactly as (A) as there are always two linearly independent (conjugate) eigenvectors:

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-.$$

However, as our dynamics take place in $X = \mathbb{R}^2$, we have to exclude non-real solutions. We know that

$$\lambda_+ = \bar{\lambda}_-, \qquad \mathbf{v}_+ = \bar{\mathbf{v}}_-,$$

where the second conjugation is taken element-wise. This compels us to require

$$C_1 = \bar{C}_2.$$

in order that all solutions be real.

Now set

$$\sigma = \Re\lambda_\pm = \frac{a+d}{2}, \qquad \tau = \pm\Im\lambda_\pm = \frac{\sqrt{|(a-d)^2+4bc|}}{2},$$

so that

$$\lambda_\pm = \sigma \pm i\tau,$$

and

$$K_1 = \frac{C_1+C_2}{2}, \qquad K_2 = i\frac{C_1-C_2}{2}.$$

We can then write $\mathbf{x}$ as

$$\mathbf{x}(t) = e^{\sigma t}\big(K_1\cos(\tau t) + K_2\sin(\tau t)\big)\Re\mathbf{v}_+ + e^{\sigma t}\big(K_2\cos(\tau t) - K_1\sin(\tau t)\big)\Im\mathbf{v}_+, \tag{9}$$

where now we can determine the constants $K_1$ and $K_2$ by the initial condition:

$$\mathbf{x}(0) = K_1\Re\mathbf{v}_+ + K_2\Im\mathbf{v}_+.$$

2.2. **Asymptotic behaviour of solutions.** In this lecture we shall look at the qualitative behaviour of solutions by looking at their behaviour asymptotically. These include behaviour near fixed points, (quasi-)periodic behaviour, and escape to infinity.

Notice that if there is a fixed point at all, i.e., if $\mathrm{d}\mathbf{x}/\mathrm{d}t = 0$, then

$$\mathbf{A}\mathbf{x} = 0,$$

which only has a trivial solution if $\mathbf{A}$ is non-singular. If $\mathbf{A}$ is singular, then 0 is an eigenvalue, and any multiple of the eigenvector(s) is a fixed point. Therefore the fixed points of these systems are either $\mathbf{0}$, or a one-dimensional subspace, or the entire $\mathbb{R}^2$. Obviously the last possibility only occurs if $\mathbf{A}$ is the zero matrix itself.

We look again at the three cases we derived for the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}(t).$$

These concerned the eigenvalues of the matrix foregoing,

$$\lambda_\pm = \frac{1}{2}\big((a+d) \pm \sqrt{(a-d)^2+4bc}\big),$$

and were:

(A) Two distinct real roots — $(a-d)^2 > -4bc$;
(B) Root with multiplicity — $(a-d)^2 = -4bc$ — this can only happen if $bc \le 0$;
(C) Conjugate roots – $(a-d)^2 < -4bc$.

Let us consider these cases in the asymptotic regime — when $t \to \infty$.

*The case (A).*

We already derived in (7) that solutions satisfy

$$\mathbf{x}(t) = C_1 e^{\lambda_+ t}\mathbf{v}_+ + C_2 e^{\lambda_- t}\mathbf{v}_-,$$

where the constants are determined by the initial condition

$$\mathbf{x}(0) = C_1\mathbf{v}_+ + C_2\mathbf{v}_-.$$

As defined, in this case, $\lambda_- < \lambda_+$.

We see that

(i) if $\lambda_- < \lambda_+ < 0$, then eventually $\mathbf{x}(t) \to \mathbf{0}$;

(ii) if $\lambda_- < 0 < \lambda_+$, then eventually $\mathbf{x}(t) \cdot \mathbf{v}_- \to 0$ ($\mathbf{x}(t)$ tends to zero in the direction of $\mathbf{v}_-$), and simultaneously, $\mathbf{x}(t) \cdot \mathbf{v}_+ \to \infty$;
(ii) if $0 < \lambda_- < \lambda_+$, then eventually $\mathbf{x}(t) \to \infty$ in the direction of $\mathbf{v}_+$.

*The case (B).*

When the geometric multiplicity of $\mathbf{A}$ equals its algebraic multiplicity, we may proceed as in the case above. The asymptotic behaviour exhibited shall either be exponential decay to the fixed point $\mathbf{0}$, or blow-up to infinity, according as the single eigenvalue-with-multiplicity satisfies $\lambda < 0$ or $\lambda > 0$.

As derived in (8), when $\mathbf{A}$ is non-diagonalizable, the solutions are

$$\mathbf{x}(t) = f(t)\mathbf{v}_1 + g(t)\mathbf{v}_2 = (C_1 + C_2 t)e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2,$$

where $\mathbf{v}_1$ is one eigenvector associated with $\lambda$ and $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$. Asymptotically, then, $\mathbf{x}$ tends to infinity or $\mathbf{0}$ in the direction of $\mathbf{v}_1$ according as $\lambda > 0$ or $\lambda < 0$.

*The case (C).*

Recall that we set $\sigma = \Re\lambda_\pm$ and $\tau = \pm\Im\lambda_-$. We derived in (9) that

$$\mathbf{x}(t) = e^{\sigma t}\big(K_1\cos(\tau t) + K_2\sin(\tau t)\big)\Re\mathbf{v}_+ + e^{\sigma t}\big(K_2\cos(\tau t) - K_1\sin(\tau t)\big)\Im\mathbf{v}_+,$$

where now we can determine the constants $K_1$ and $K_2$ by the initial condition:

$$\mathbf{x}(0) = K_1\Re\mathbf{v}_+ + K_2\Im\mathbf{v}_+.$$

These solutions are periodic/oscillatory with an attenuation/damping or amplification coefficient $e^{\sigma t}$. It is an attenuation factor if $\sigma < 0$ whereupon asymptotically, the solution tends to $\mathbf{0}$. It is an amplification factor if $\sigma > 0$, and asymptotically, the solution tends to infinity in an oscillatory manner. If $\sigma = 0$, then the oscillatory/periodic behaviour persists for all time.

2.3. **A word on non-homogeneity.** Only a slight modification is needed to treat systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g},$$

where $\mathbf{g} \in \mathbb{R}^d$ if zero is not an eigenvalue of $\mathbf{A}$ (that is, if $\mathbf{A}$ is invertible). If $\mathbf{A}$ is invertible, we can find a unique vector $\mathbf{h} \in \mathbb{R}^d$ such that $\mathbf{A}\mathbf{h} = \mathbf{g}$. Then using the change-of-variable $\mathbf{y} = \mathbf{x} + \mathbf{h}$ — a constant translation, we can write the system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t).$$

We can solve for $\mathbf{y}$ in the manner already prescribed (by exponentiation of $\mathbf{A}$), and then recover $\mathbf{x}$ via $\mathbf{x} = \mathbf{y} - \mathbf{h}$. The only difference then is that, in the case $d = 2$ for example, the fixed points are now either at $\mathbf{h}$.

If $\mathbf{A}$ is singular, assume that it has been reduced to Jordan normal form. Then either

$$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Both of these cases can be solved directly.

2.4. **Phase Portraits.** As described in Lecture 1, for $d = 2$, there is a particular vivid way to represent solutions via phase portraits. By definition, for autonomous systems, the flow $\phi(t, s, \mathbf{x}(0))$ only depends on $t$ and $s$ via the duration $t - s$. Therefore the (forward) orbits

$$\Gamma^+(\mathbf{x}(0)) = \{\phi_t(\mathbf{x}(0)) : t \in \mathbb{R}\}$$

form a set partially ordered by inclusion — for any two initial points $\mathbf{x}(0)$ and $\mathbf{y}(0)$, either

(i) $\Gamma^+(\mathbf{x}(0)) \subseteq \Gamma^+(\mathbf{y}(0))$, or
(ii) $\Gamma^+(\mathbf{y}(0)) \subseteq \Gamma^+(\mathbf{x}(0))$, or
(iii) $\Gamma^+(\mathbf{x}(0)) \cap \Gamma^+(\mathbf{y}(0)) = \varnothing$.

A cleaner but slightly coarser statement is that the orbits

$$\Gamma(\mathbf{x}(0)) = \{\phi_t(\mathbf{x}(0)) : t \in \mathbb{R}\}$$

either coincide or are disjoint. This observation ensures that we can draw reasonably clean phase portraits for 2-dimensional linear autonomous systems. We shall now devote the remainder of this lecture to sketching phase portraits of each of the cases mentioned in the previous subsection.

orbits : $\mathcal{O}(x(0)) = \{ \phi_t(x(0)) : t \in \mathbb{R} \}$

~~are~~ either coincide or are disjoint.

(A) i) Two distinct real eigenvalues

$0 < \lambda_1 < \lambda_2$

$$\dot{x} = 2x + 2y$$
$$\dot{y} = -x + 5y \qquad \underline{\underline{A}} = \begin{pmatrix} 2 & 2 \\ -1 & 5 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

find eigenvalues:

$$0 = \det(A - \lambda I) = (\lambda - 2)(\lambda - 5) + 2 = (\lambda - 3)(\lambda - 4$$

eigenvectors:

$\lambda = 3$

$$\begin{pmatrix} 2-3 & 2 \\ -1 & 5-3 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = 4$

$$v_2 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

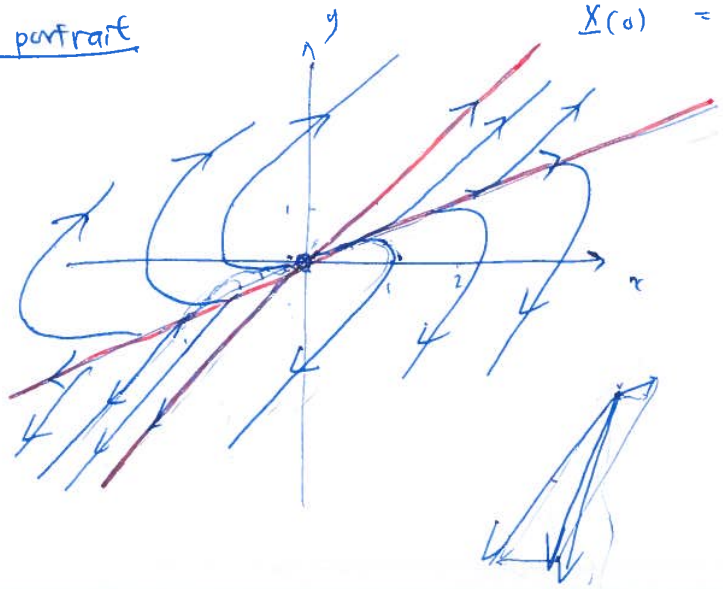


general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underline{x}(t) = C_1 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

phase portrait

$$\underline{x}(0) = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$




UNSTABLE NODE

~~FOCUS~~

(A) (i) $\lambda_1 < \lambda_2 < 0$

$$\dot{x} = -3x + y \quad \dot{y} = x - 3y \quad \rightarrow \quad \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

finding eigenvalues

$$0 = \det(A - \lambda I) = (\lambda+2)(\lambda+4)$$

finding eigenvectors

$\lambda = -2$

$$\begin{pmatrix} -3+2 & -1 \\ 1 & -3+2 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 0 \quad \rightarrow \quad v_1 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector}$$

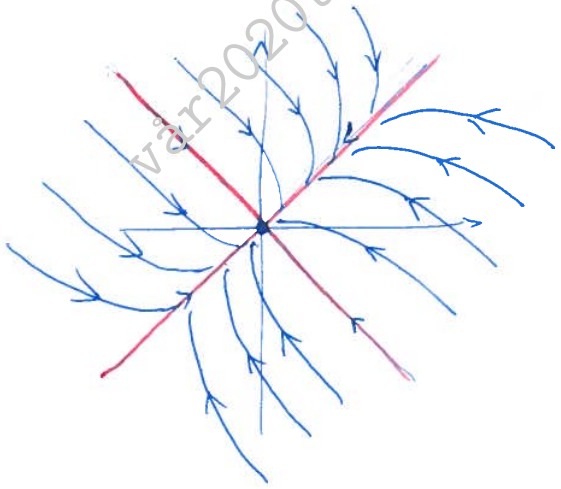
$\lambda = -4$

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector.}$$

general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

phase portrait



STABLE NODE

(iii) $\lambda_1 < 0 < \lambda_2$ $\quad \dot{x} = x - 2y$, $\dot{y} = -3x + 2y$

$\hookrightarrow A = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$

$0 = \det(A - \lambda I) = (\lambda - 4)(\lambda + 1)$

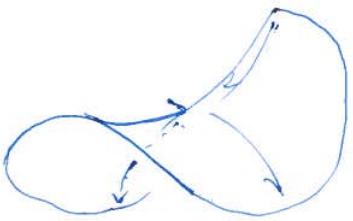
$\underline{\lambda = 4}$ $\quad v_1 = \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

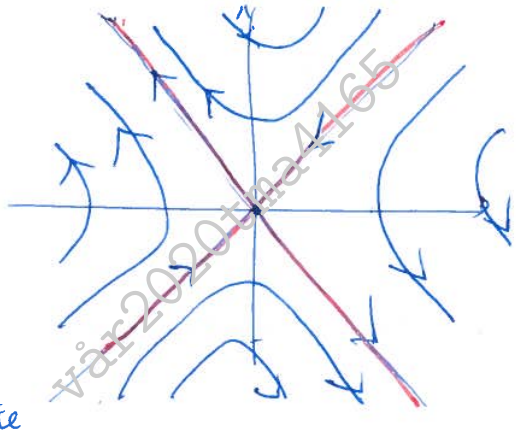
$\underline{\lambda = -1}$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = C_1 e^{4t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

phase portrait





SADDLE



(B) Root with degenerate multiplicity $\lambda \neq 0$

$\dot{x} = x + y$, $\dot{y} = y$ $\quad \rightarrow A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$0 = \det(A - \lambda I) = (\lambda - 1)^2$.

~~Jordan~~ eigenvector: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

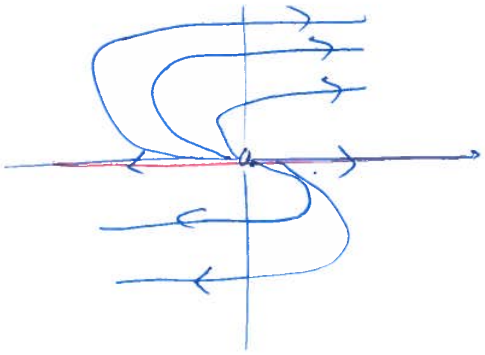
Jordan chain.

$(A - \lambda I) v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (C_1 + C_2 t) e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



when $x=0$, $y>0$

$\dot{x}>0$

when $x=0$, $y<0$

$\dot{x}<0$

DEGENERATE UNSTABLE NODE

(c) Conjugate roots

$\Re\lambda > 0$

$$\dot{x} = 3x - 2y$$
$$\dot{y} = 2x + 3y \longrightarrow A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

finding eigenvalues

$$0 = \det(A - \lambda I) = (\lambda - (3+2i))(\lambda - (3-2i))$$

"eigenvectors"

$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (3+2i)\begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

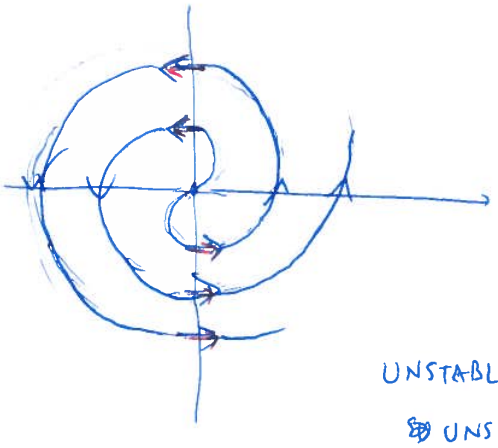
$\longrightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector $(\mathbb{C}^2 \longrightarrow \mathbb{C}^2)$

general solution

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = e^{3t}(K_1\cos(2t) + K_2\sin(2t))\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Re\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$+ e^{3t}(K_2\cos(2t) - K_1\sin(2t))\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \Im\begin{pmatrix} 1 \\ i \end{pmatrix}$$



phase portrait



when $x=0$, $y>0$

$\longrightarrow \dot{x} < 0$

when $x=0$, $y<0$

$\longrightarrow \dot{x} > 0$

UNSTABLE FOCUS/

UNSTABLE SPIRAL

③ Conjugate roots
$\Re\lambda > 0$

$$\dot{x} = 3x - 2y$$
$$\dot{y} = 2x + 3y \quad \rightarrow \quad \underline{\underline{A}} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$$
$$= (\lambda - 3)^2 + 4$$
$$= (\lambda - (3+2i))(\lambda - (3-2i))$$

"eigenvectors".

$$\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (3+2i) \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

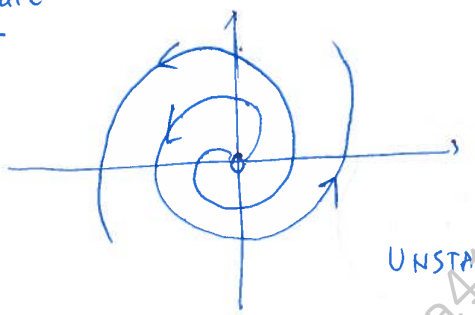
↖ [we know the "other eigenvector" already — by conjugating the entire equation]

$\rightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an "eigenvector".

general solution :

$$\underline{x}(t) = e^{3t}(K_1 \cos(2t) + K_2 \sin(2t)) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t}(K_2 \cos(2t) - K_1 \sin(2t)) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

phase portrait



when $x = 0$, $y > 0$, $\dot{x} < 0$

when $y = 0$, $x > 0$, $\dot{y} > 0$

UNSTABLE SPIRAL / FOCUS

④ Conjugate Roots
$\Re\lambda < 0$

$$\dot{x} = -5x - 5y$$
$$\dot{y} = 2x - y \quad \rightarrow \quad \underline{\underline{A}} = \begin{pmatrix} -5 & -5 \\ 2 & -1 \end{pmatrix}$$

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$$
$$= (\lambda + 5)(\lambda + 1) + 10$$
$$= (\lambda - (-2+i))(\lambda + (-2-i))$$

"eigenvectors"

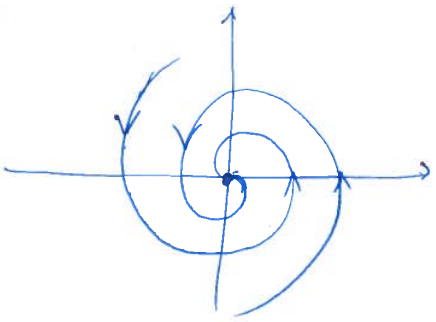
$$\begin{pmatrix} -5 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = (-2+i) \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} -2-i \\ 1+i \end{pmatrix}$ is an "eigenvector"

general solution :

$$\underline{x}(t) = e^{-2t}(K_1 \cos(t) + K_2 \sin(t)) \begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^{-2t}(K_2 \cos(t) - K_1 \sin(t)) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

phase portrait :



when $x = 0$, $y > 0$, $\dot{x} < 0$

STABLE SPIRAL / FOCUS

(C) a "degenerate" example

conjugate roots

$\Re\lambda = 0$.

$\dot{x} = 2x + 4y$

$\dot{y} = -5x - 2y$ $\longrightarrow$ $A = \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}$

$0 = \det(A - \lambda I)$

$= (\lambda^2 + 16)$

$= (\lambda - 4i)(\lambda + 4i)$

"eigenvectors"

$\begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = 4i \begin{pmatrix} r^1 \\ r^2 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} r^1 \\ r^2 \end{pmatrix} = \begin{pmatrix} 2 - 4i \\ 3 + 4i \end{pmatrix}$ is an "eigenvector".

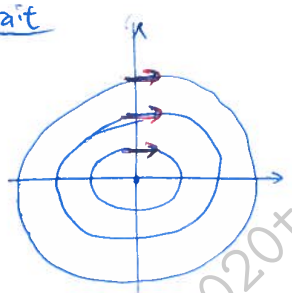
general solution

$= \Re \begin{pmatrix} 2 - 4i \\ 3 + 4i \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}(t) = (K_1 \cos(4t) + K_2 \sin(4t)) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$+ (K_2 \cos(4t) - K_1 \sin(4t)) \begin{pmatrix} -4 \\ 4 \end{pmatrix}$ $= \Im \begin{pmatrix} 2 - 4i \\ 3 + 4i \end{pmatrix}$

phase portrait



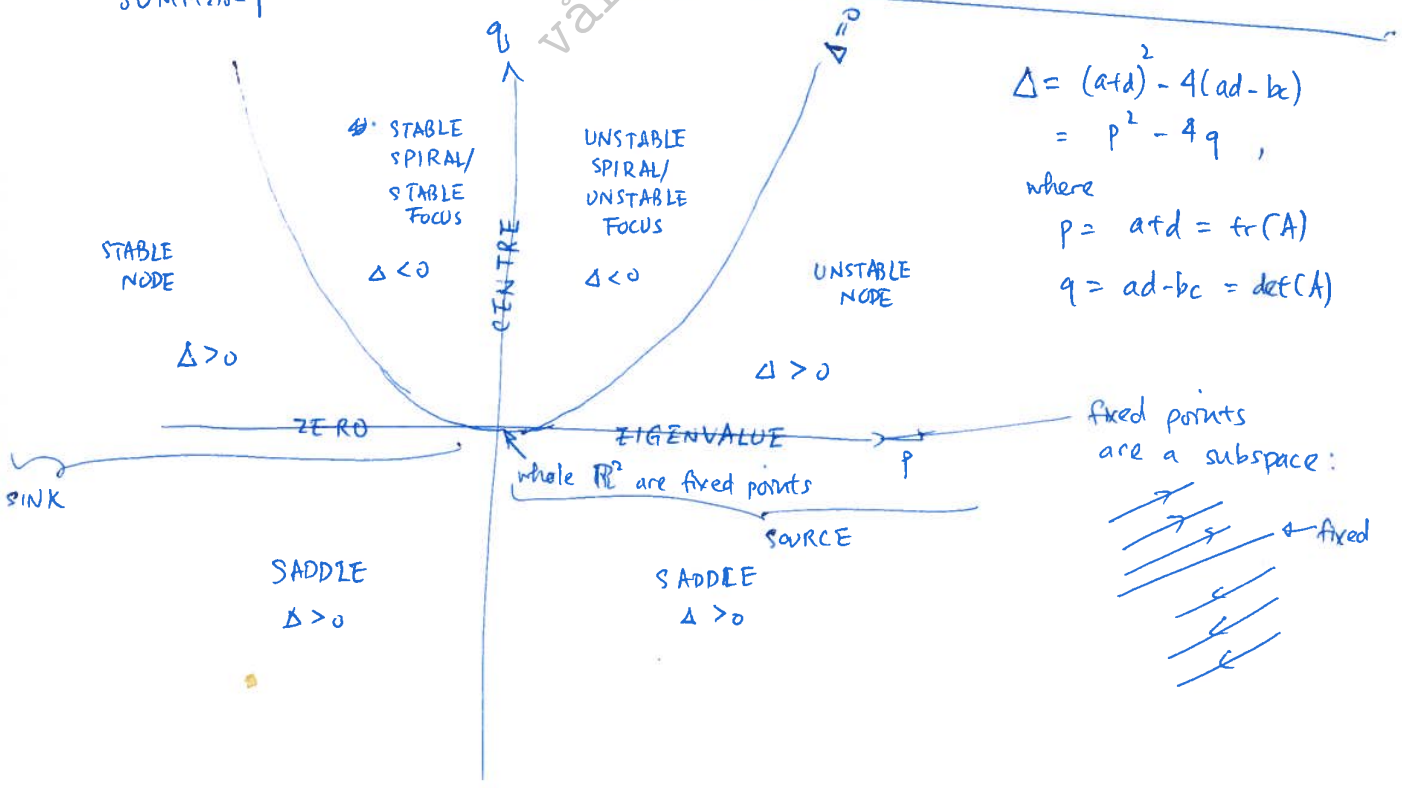
when $x = 0$, $y > 0$.

$\longrightarrow \dot{x} > 0$

CENTRE



SUMMARY




$\Delta = (a+d)^2 - 4(ad - bc)$

$= p^2 - 4q$,

where

$p = a + d = \mathrm{tr}(A)$

$q = ad - bc = \det(A)$

fixed points are a subspace:

## 3. Lecture III: Linear Systems on $\mathbb{R}^d$

In this lecture, we look at linear systems more generally.

### 3.1. Autonomous linear systems on $\mathbb{R}^d$.

We consider again the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t),$$

where $\mathbf{A}$ is now a $d \times d$ matrix. Now as we saw, the matrix $\mathbf{A}$ is not in general diagonalizable. However, it can always be put into a standard, or *normal* form, from which we can determine which eigenvalues have strictly greater algebraic multiplicities than they have geometric multiplicities. We have the following useful theorem:

**Theorem 3.1** (Jordan's Normal Form Theorem)**.** *Let $\mathbf{B}_m$ for $1 \leq m \leq s$ be a $d_m \times d_m$ matrix given by*

$$\big(\mathbf{B}_m\big)_i^j = \lambda_m \delta_i^j + \delta_{i-1}^j.$$

*For every $d \times d$ matrix $\mathbf{A}$, there exist $d_m \times d_m$ matrices $\mathbf{B}_m$ with $\sum_{1\leq m\leq s} d_m = d$, and a $d \times d$ matrix $\mathbf{J}$ given by*

$$\mathbf{J} = \begin{pmatrix} \mathbf{B}_1 & & \\ & \ddots & \\ & & \mathbf{B}_s \end{pmatrix}$$

*for which*

$$\mathbf{A} \sim \mathbf{J}.$$

*The matrix $\mathbf{J}$ is unique up to the ordering of the blocks.*



Recall that similarity is an equivalence relation.

The matrix $\mathbf{J}$ is the Jordan normal form of $\mathbf{A}$. We sometimes write this as $\mathbf{J} = \mathrm{diag}(\mathbf{B}_m)$. The matrices $\mathbf{B}_m$ with $\lambda_m$ on the diagonal and 1 on the superdiagonal are known as the Jordan blocks. The number of Jordan blocks with $\lambda$ on the diagonal is the geometric multiplicity of that eigenvalue and the number of times $\lambda$ appears in $\mathbf{J}$ itself is its algebraic multiplicity.

As $\mathbf{A}$ is real, its characteristic equation also only has real coefficients, and therefore complex roots only appear in conjugate pairs.

This normal form representation allows us to decouple the first-order system into blocks with distinct eigenvalues of different multiplicities. We shall now repeat in general dimensions our deductions in the case $d = 2$. In particular, from the Fundamental Theorem for Linear Systems (Thm.1.2) it can be seen (cf. (6)) that where $\mathbf{A} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$,

$$\mathbf{x}(t) = \mathbf{P} \exp\big(\mathbf{J}t\big)\mathbf{P}^{-1}\mathbf{x}(0).$$

One advantage of decoupling the equations is that, just as diagonal matrices are easy to exponentiate, it is also relatively benign a calculation that determines $\exp\big(\mathbf{J}t\big)$. First, a direct calculation shows that for any power $n \in \mathbf{N}$,

$$\mathbf{J}^n = \begin{pmatrix} \mathbf{B}_1^n & & \\ & \ddots & \\ & & \mathbf{B}_s^n \end{pmatrix},$$

so that by the series representation of the exponential,

$$\exp\big(\mathbf{J}t\big) = \exp\big(\mathrm{diag}(\mathbf{B}_m t)\big) = \mathrm{diag}\big(\exp(\mathbf{B}_m t)\big).$$

Next it remains to find $\exp\big(\mathbf{B}_m t\big)$. Writing $\mathbf{N}_m$ for the $d_m \times d_m$ matrix with 1s along the superdiagonal (with entries $\delta_{i-1}^j$), we find

$$\exp(\mathbf{B}_m) = \exp(\lambda_m \mathbf{I}_{d_m} t)\exp(\mathbf{N}_m t) = e^{\lambda_m t}\exp(\mathbf{N}_m t),$$

as the identity matrix commutes with any other matrix.

Therefore we need only determine $\exp(\mathbf{N}_m t)$. We find that $\mathbf{N}_m$ is nilpotent with power $d_m$, as, by direct calculation (or by induction), for any power $n \in \mathbb{N}$,

$$\left(\mathbf{N}_m^n\right)_i^j = \delta_{i-n}^j.$$

Putting this into the series expansion for $\exp(\mathbf{N}_m t)$ we find

$$\Big(\exp(\mathbf{N}_m t)\Big)_i^j = \sum_{n=0}^{d_m-1} \frac{t^n}{n!}\delta_{i-n}^j,$$

i.e., $\exp(\mathbf{N}_m t)$ is an upper-diagonal matrix that looks like

$$\exp(\mathbf{N}_m t) = \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{d_m-1}/(d_m-1)! \\ 0 & 1 & t & \cdots & t^{d_m-2}/(d_m-2)! \\ 0 & 0 & 1 & \cdots & t^{d_m-3}/(d_m-3)! \\ \cdots & & & & \\ 0 & \cdots & & 1 & t \\ 0 & \cdots & & 0 & 1 \end{pmatrix}.$$

Recall that the algebraic multiplicity of an eigenvalue $\lambda$ is the number of times that $\lambda$ is a root of $p(x) = \det(\mathbf{A} - x\mathbf{I}) = 0$, and the geometric multiplicity is the dimension of $\ker(\mathbf{A} - \lambda\mathbf{I})$.

Associated with each *distinct* eigenvalue $\lambda_n$ of $\mathbf{A}$, of algebraic multiplicity $\mu_n$ is a generalized eigenspace spanned by the $\mu_n$ linearly independent generalized eigenvectors $V_n = \{\mathbf{v}_n^1, \ldots, \mathbf{v}_n^{\mu_n}\}$. Of these, a subset will be actual eigenvectors. The rest are found by the Jordan chain procedure. The Jordan chain procedure by which the bases $V_n$ are found is given in *Perko* on p.43. Note that the algorithm/theorem in Perko is written in such a way as to avoid vectors with complex entries.

The idea is that if the dimension of $\ker(\mathbf{A}-\lambda\mathbf{I})$ is not great enough, we look among $\ker(\mathbf{A}-\lambda\mathbf{I})^2$, and increasingly higher powers, until we find $d$ linearly independent vectors.

The generalized eigenvectors give the full generalized basis by which $\mathbf{P}$ is constructed so that $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$.



**Example 3.1.** Examples of matrices in Jordan normal form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Example 3.2.** Consider the Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{b}.$$

Let the matrix be called $\mathbf{A}$.

One can find that the eigenvalues of the matrix are $\lambda_1 = 2$, with algebraic multiplicity 2, and $\lambda_2 = 3$.

However, $\lambda_1$ only has geometric multiplicity of 1, and its associated eigenvector is $\mathbf{v}_1 = (1,0,0)^\top$.

The eigenvector associated with $\lambda_2$ is $\mathbf{v}_2 = (0,0,1)^\top$.

This tells us that the Jordan normal form is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We can also find the non-singular matrices $\mathbf{P}$ that puts $\mathbf{A}$ into its Jordan normal form. The Jordan chain procedure involves finding an linearly independent vector $\mathbf{v}_3$ in the kernel of $(\mathbf{A} - \lambda_1\mathbf{I}_3)^2$ as

$\ker(\mathbf{A} - \lambda\mathbf{I}_3)$ has been exhausted. This is done by setting

$$(\mathbf{A} - \lambda_1\mathbf{I}_3)\mathbf{v}_3 = \mathbf{v}_1.$$

Doing so we find that $\mathbf{v}_3 = (0, 1, 1)^\top$ is a solution.

The subspace decomposition then *suggests* that

$$\mathbf{x}(t) = (C_1 + C_2 t)e^{2t}\mathbf{v}_1 + C_2 e^{2t}\mathbf{v}_3 + C_3 e^{3t}\mathbf{v}_2.$$

The reason for finding extra linearly independent vectors to be generalized eigenvectors in this way is that, setting

$$\mathbf{P} = (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

we have

$$\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}.$$

Therefore with the change-of-variables $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$, we can transform the Cauchy problem to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{J}\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{P}^{-1}\mathbf{b}.$$

This can be exponentiated easily:

$$\mathbf{y}(t) = (C_1 + C_2 t)e^{2t}\mathbf{P}^{-1}\mathbf{v}_1 + C_2 e^{2t}\mathbf{P}^{-1}\mathbf{v}_3 + C_3 e^{3t}\mathbf{P}^{-1}\mathbf{v}_2,$$

(because if $\mathbf{w}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{A}\mathbf{P}\mathbf{P}^{-1}\mathbf{w} = \lambda\mathbf{w}$, and $\mathbf{P}^{-1}\mathbf{w}$ is the eigenvector of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$).

We can recover $\mathbf{x}$ via $\mathbf{x} = \mathbf{P}\mathbf{y}$, verifying our intuitions.

3.2. **Linear stability theory.** The main advantage of decoupling the system by casting it into the Jordan normal form, however, is that we can state a sort of "structure theorem" for all linear systems.



Let $V_n$ be the collection of generalized eigenvectors associated with the eigenvalue $\lambda_n \in \mathbb{C}$. We can group the collections $V_n$ into three classes: Set

$$\begin{aligned} E^s &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n<0\}} V_n \\ E^c &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n=0\}} V_n \\ E^u &= \operatorname{span} \bigcup_{\{n:\Re\lambda_n>0\}} V_n. \end{aligned} \tag{10}$$

We call $E^s$ the STABLE SUBSPACE, $E^c$ the CENTRE SUBSPACE, and $E^u$ the UNSTABLE SUBSPACE. They are so named because $E^s$ consist of the directions along which the dynamics of the system enforces decay to $\mathbf{0}$ and $E^u$ consist of the directions along which $\mathbf{x}(t)$ tends to infinity as $t \to \infty$.

We say that a subspace $E \subseteq \mathbb{R}^n$ is INVARIANT with respect to the flow $\phi_t$ if $\phi_t E \subseteq E$ for all time $t \in \mathbb{R}$. That is, the (forward and backward) orbit(s) of each point within $E$ remains in $E$.

For us, the flow, as identified following (5) is $\exp(\mathbf{A}t)$. If all eigenvalues of $\mathbf{A}$ have non-zero real parts (i.e., the centre manifold is trivial), then we say that the flow, and the linear system, is HYPERBOLIC.

From the definition of the generalized eigenvectors that span $E^s$, $E^c$, and $E^u$, we can deduce that:

**Lemma 3.2.** *Let $E$ be the generalized eigenspace of $\mathbf{A}$ corresponding to an eigenvalue $\lambda$. Then $\mathbf{A}E \subseteq E$.*

Let $\hat{E}$ be the subspace of the generalized eigenvectors $E$ that is spanned by the eigenvectors of $\mathbf{A}$. This lemma boils down to the fact that if $\mathbf{u} \in E$, then it can be written as a linear combination of eigenvectors $\mathbf{e}_i$ or generalized eigenvectors $\mathbf{v}_i$, for which $\mathbf{A}^k \mathbf{v}_i \in \hat{E}$ for high enough $k$. Now for every $k$ , including $k = 1$, $\mathbf{v}_i$ being defined using a Jordan chain, must still satisfy that $\mathbf{A}\mathbf{v}_i \in \lambda \mathbf{v}_i + E$. And for eigenvectors $\mathbf{e}_i$, it is even better — $\mathbf{A}\hat{E} = \hat{E}$, unless 0 is an eigenvalue, in which case the equality again reverts to an inclusion.

Since $\mathbf{A}$ acts linearly over the linear combination, $\mathbf{A}E \subseteq E$.

Iterating the theorem we see that

$$\mathbf{A}^n E \subseteq \mathbf{A}^{n-1} E \subseteq \ldots \subseteq \mathbf{A}E \subseteq E.$$

So using the partial sums of series representation of the flow $\exp(\mathbf{A}t)$, and taking the limit, it further holds that the subspaces are invariant with respect to the flow also. This invariance means that in general, the Jordan normal form decouples the system into invariant subspaces, and these invariant subspaces span the entire phase space:

**Theorem 3.3.** *Let* $\mathbf{A}$ *be a real* $d \times d$ *matrix. Then the phase space* $\mathbb{R}^d$ *can be decomposed thus:*

$$\mathbb{R}^d = E^s \oplus E^c \oplus E^u,$$

*which are defined in* (10)*. Furthermore, these subspaces are each invariant with respect to the flow* $\exp(\mathbf{A}t)$*.*

3.3. **A class of nonautonomous linear systems.** Finally we take a look at a simple class of *non*-autonomous systems, which is the autnomous linear system, with an additional nonhomogeneity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t). \tag{11}$$

From a first course in differential equations, it can be guessed that the solution is the matrix reformulation of Duhamel's formula, by obtained by convolving a fundamental solution against the inhomogeneity. This is in fact the case.

Define a FUNDAMENTAL MATRIX SOLUTION to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

as any continuously differentiable matrix-valued function $\Phi : t \to \mathbb{R}^{d\times d}$ satisfying

$$\Phi'(t) = \mathbf{A}\Phi(t) \qquad \forall t \in \mathbb{R}. \tag{12}$$

In particular, $\Phi(t) = \exp(\mathbf{A}t)$ is a fundamental matrix solution of the first-order system with $\Phi(0) = \mathbf{I}_d$. We have the theorem:

**Theorem 3.4.** *Let $\Phi$ be any fundamental matrix solution to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t).$$

*The solution of (11) with intitial condition $\mathbf{x}(0) = \mathbf{b}$ can be written as*

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{b} + \int_0^t \Phi(t)\Phi^{-1}(s)\mathbf{g}(s)\,\mathrm{d}s,$$

*and is unique.*

This result can be verified readily by differentiation.

This also means that taking $\Phi(t) = \exp(\mathbf{A}t)$, we have

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{b} + \int_0^t \exp(\mathbf{A}(t-s))\mathbf{g}(s)\,\mathrm{d}s.$$

It bears mentioning that, in fact, we can always turn a non-autonmous system into an autonmous one by increasing our dimension. Suppose we have some non-autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)).$$

Now introduce a new variable $\mathbf{y}(t) = (\tau(t), \mathbf{x}^\top(t))^\top$, where $\tau(t) = t$. Then the system becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau = 1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{y}(t)),$$

which is obviously autonomous. However, even if $f$ were linear in $\mathbf{x}(t)$, it is usually very far from being linear in $\mathbf{y}(t)$.

## 4. Lecture IV: Local Well-Posedness I

The study of differential equations is usually concerned with four main problems. Given a differential equation, we should like to know:

(i) Existence: given an appropriate set of initial/boundary conditions, do solutions exist (in a certain function space, in an appropriate sense)?
(ii) Uniqueness: given an appropriate set of initial/boundary conditions, are solutions unique if they exist (in an appropriate sense)?
(iii) Continuous dependence: if initial/boundary/other conditions are perturbed slightly (in a given topology), does the corresponding solution also only change slightly (in some topology)? (Continuous dependence implies uniqueness in appropriate spaces.)
(iv) Asymptotic/long-time behaviour: concerning evolution equations — how do solutions develope eventually ?

Along the way, we are interested also in dependence on other parameters, transient behaviour of solutions, etc.. Affirmative answers to the first three questions characterize a phenomenon known as well-posedness. We say the equation with its accompanying conditions constitute then a "well-posed problem".

In previous lectures, we have answered all of the questions above as they concern linear autonomous systems of two first-order equations by constructing solutions explicitly. Direct construction is not generally possible. We should like to provide some general conditions for well-posedness of first-order systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)).$$



To this end we shall establish the Picard-Lindelöf Theorem in this lecture about existence and uniqueness, and we shall discuss continuous dependence in Lecture 5. Before we do so we shall recall an important definition: A function $g : \mathbb{R}^m \to \mathbb{R}^d$ is Lipshitz in the open set $U \subseteq \mathbb{R}^m$ if there is a constant $L$, known as the Lipschitz constant, such that for every $\mathbf{x}, \mathbf{y} \in U$,

$$|g(\mathbf{x}) - g(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

Clearly, Lipschitz functions are continuous, and in fact, they are differentiable except on a Lebesgue null set. Where a Lipschitz function is differentiable, its derivative is readily seen to be bounded by the local Lipschitz constant. Therefore Lpischitzness is a smoothness condition. When $L < 1$ globally, we call the Lipschitz map a contraction map.

This condition is important in ways we shall appreciate later. First we take a look at two classic examples of non-(global) existence and non-uniqueness where the Lipschitz condition is violated:

**Example 4.1.** On the space $X = \mathbb{R}$ consider the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \qquad x(0) = 1.$$

It can be checked that $x \mapsto x^2$ fails to be globally Lipschitz, though it is locally so on every bounded open set. This equation is separable, and we can solve it to find

$$\frac{\mathrm{d}x}{x^2} = \mathrm{d}t, \qquad \frac{-1}{x} = t + C.$$

The constant is $-1$ from the initial condition and we find that the solution is

$$x(t) = \frac{1}{1-t}.$$

There is no existence beyond $[0, 1)$. We call this mode of non-existence finite-time blow-up.

**Example 4.2.** Again consider a one-dimensional system this time given by the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{|x|}, \qquad x(0) = 0.$$

It can be checked that $x \mapsto \sqrt{|x|}$ fails to be Lipschitz in any neighbourhood of 0.

The equation is again separable, and we find that

$$x(t) = t^2/4$$

solves the problem with the given initial condition. But this initial condition is not enough to enforce uniqueness because $x(t) \equiv 0$ also solves the equation with the given initial condition.

We are now ready to state the main theorem of this lecture.

**Theorem 4.1** (Picard-Lindelöf Theorem)**.** *Let $f(t, \mathbf{x}(t))$ be Lipschitz in its second argument over the open set $U \subseteq \mathbb{R}^d$ and continuous in $t$. Then for each $\mathbf{b} \in \mathbb{R}^d$, there exists an $\eta > 0$, and a $C^1$ map $\mathbf{x} : (t_0 - \eta, t_0 + \eta) \to U$ solving the Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)), \qquad \mathbf{x}(t_0) = \mathbf{b}. \tag{13}$$

*Furthermore, $\mathbf{x}$ is unique on its interval of definition.*

Note that this theorem extends the theorem in *Cain and Schaeffer* to the non-autonomous case.

The proof depends on several lemmas and is, along with the proofs of these lemmas, examinable. The proof is bipartite.

(1) First we shall show that the Cauchy problem for the differential equation can be re-formulated as a Cauchy problem for an integral equation. Simultaneously, we shall show that if a continuous $\mathbf{x}(t)$ exists at all, it is in fact $C^1$. This shall re-cast our problem as a FIXED-POINT PROBLEM.

(2) Secondly, we shall show that $\mathbf{x}(t)$ exists and is continuous and unique.

*Part 1.*

**Lemma 4.2.** *Let $U \subseteq \mathbb{R}^d$ be an open set. Let $f : \mathbb{R} \times U \to \mathbb{R}^d$ be continuous. Write $J$ for the interval $(t_0 - \eta, t_0 + \eta)$. If $\mathbf{x} \in C^1(J)$ solves the Cauchy problem (13) then it also solves the integral equation*

$$\mathbf{x}(t_0 + t) = \mathbf{x}(t_0) + \int_{t_0}^{t_0+t} f(s, \mathbf{x}(s))\, \mathrm{d}s, \qquad 0 < t < \eta. \tag{14}$$

*Conversely, if $\mathbf{x} \in C(J)$ and satisfies the foregoing integral equation, then it is in fact in $C^1(J)$ and solves the Cauchy problem (13).*



This is a direct result of an application of the fundamental theorem of calculus. Let it be pointed out that $\mathbf{x}(t)$ needs only be continuous for the integral equation to make sense. But once it does make sense, it is immediate that $\mathbf{x}(t)$ must also be once continuously differentiable if $f$ is continuous in both its arguments as a composition of two continuous functions is continuous.

This transforms our Cauchy problem into one of finding a fixed point $\mathbf{x} \in C(J)$ (the "point" is a continuous function — a point on the function space) in the following way. Let $\mathfrak{T} : C(J) \to C(J)$ via

$$\mathfrak{T}(\mathbf{x})(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}(s))\, \mathrm{d}s.$$

Then the Cauchy problem is equivalent to finding a fixed point of the map $\mathfrak{T}$.

*Part 2.*

Establishing existence usually turns out to be a topological result of compactness, or on a metric space, completeness. We know that a complete space is one for which every Cauchy sequence converges. That is, roughly if a sequence does not escape to infinity — that a subsequence is Cauchy — then the space ensures that it also does not eventually fall through a hole in the space, and so a limit point exists. Therefore existence reduces to setting up an approximate sequence and showing that the elements of the sequence gets closer and closer to one another.

We shall require then three ingredients:

(i) finding a complete metric space,
(ii) constructing a sequence,
(iii) showing that the sequence is Cauchy,

or satisfy some other compactness structure analogous to the ones here described.

Looking back at Part 1., we see that we have an essentially free upgrade to the smaller space of continuously differentiable functions, and we need only to show that our solution exists as a continuous function over some interval $J$. Therefore our candidate for a complete metric space is $C(J)$, with $\eta$ a free constant to be later determined. Obviously we should like to make $\eta$ (in $J = (t_0 - \eta, t_0 + \eta)$) as big as we can.

**Lemma 4.3.** *The continuous functions $C(J; \mathbb{R}^d)$ form a complete metric space under the norm-induced metric*

$$\|\mathbf{x} - \mathbf{y}\|_{C(J)} = \sup_{t \in J} |\mathbf{x}(t) - \mathbf{y}(t)|.$$

It is also relatively transparent that $C(J)$ is a vector space. Recall that a complete normed vector space is known as a Banach space.

*Proof.* To show completeness, we postulate a Cauchy sequence of continuous functions and show that it must converge to a continuous function. Let $\mathbf{x}_n(t)$ be a sequence of functions in $C(J)$ such that for any $\varepsilon$, there is an $N$ such that for $m, n > N$,

$$\|\mathbf{x}_n - \mathbf{x}_m\|_{C(J)} = \sup_{t \in J} |\mathbf{x}_n(t) - \mathbf{x}_m(t)| < \varepsilon.$$

For each fixed $t \in J$, given any $\varepsilon$, we can choose the same $N$ and say that if $n, m > N$,

$$|\mathbf{x}_n(t) - \mathbf{x}_m(t)| \leq \|\mathbf{x}_n - \mathbf{x}_m\|_{C(J)} < \varepsilon.$$

Therefore by the completeness of the reals, for each fixed $t$, $\mathbf{x}_n(t)$ converges pointwise and defines a point $\mathbf{x}(t)$. Now we shall show that $t \mapsto \mathbf{x}(t)$ is continuous:

For two points $s, t \in (t_0 - \eta, t_0 + \eta)$, by the triangle inequality,

$$|\mathbf{x}(t) - \mathbf{x}(s)| \leq |\mathbf{x}(t) - \mathbf{x}_n(t)| + |\mathbf{x}_n(t) - \mathbf{x}_n(s)| + |\mathbf{x}_n(s) - \mathbf{x}(s)|.$$

Let us choose $N$ large enough that $\sup_{t \in J} \mathbf{x}(t) - \mathbf{x}_n(t)| < \varepsilon/3$. Since $\mathbf{x}_n$ is continuous, for any $\varepsilon > 0$, there exists a $\delta$ such that if $|t - s| < \delta$, $|\mathbf{x}_n(t) - \mathbf{x}_n(s)| < \varepsilon/3$. The calculation above then implies that for every $\varepsilon > 0$, we can use the same $\delta$ and find that if $|t - s| < \delta$, $|\mathbf{x}(t) - \mathbf{x}(s)| < \varepsilon$. Therefore $\mathbf{x} \in C(J)$, and the space is complete.

□

The next lemma will use this completeness to find a unique solution to a fixed-point problem on the Banach space.

**Lemma 4.4** (Contraction Mapping Principle)**.** *Let $\mathfrak{T} : \mathfrak{X} \to \mathfrak{X}$ be a contraction map from a Banach space into itself. Then $\mathfrak{T}$ has a unique fixed point. that is, there exists a unique $\mathbf{x} \in \mathfrak{X}$ such that*

$$\mathfrak{T}(\mathbf{x}) = \mathbf{x}.$$

*Proof.* By assumption, $\mathfrak{T}$ is a contraction map. This means that for some $L < 1$,

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} \leq L\|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}$$

for any two continuous functions $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$.

Define the iteration $\mathbf{x}_{n+1} = \mathfrak{T}(\mathbf{x}_n)$.

Then we find that

$$\|\mathfrak{T}(\mathbf{x}_{n+1}) - \mathfrak{T}(\mathbf{x}_n)\|_{\mathfrak{X}} \leq L\|\mathfrak{T}(\mathbf{x}_n) - \mathfrak{T}(\mathbf{x}_{n-1})\|_{\mathfrak{X}} \leq L^{n+1}\|\mathfrak{T}(\mathbf{x}_0) - \mathbf{x}_0\|_{\mathfrak{X}}.$$

Since $L^n \to 0$,

$$\|\mathbf{x}_m - \mathbf{x}_n\|_{\mathfrak{X}} \leq \sum_{k=m+1}^{n+1} L^k\|\mathfrak{T}(\mathbf{x}_0) - \mathbf{x}_0\|_{\mathfrak{X}},$$

and $\{\mathbf{x}_n\}$ is a Cauchy sequence. By assumption the space $\mathfrak{X}$ is Banach and hence complete. Therefore, there is a point $\mathbf{x}$ to which the sequence converges in the norm of $\mathfrak{X}$. This must be a fixed point by construction.

Suppose there are two fixed points $\mathbf{x}$ and $\mathbf{y}$. Then

$$\mathfrak{T}(\mathbf{x}) = \mathbf{x}, \qquad \mathfrak{T}(\mathbf{y}) = \mathbf{y}, \qquad \|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} = \|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}.$$

But this violates the contraction property as

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{\mathfrak{X}} \leq L\|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}} < \|\mathbf{x} - \mathbf{y}\|_{\mathfrak{X}}.$$

This contradiction establishes the theorem.

□



*Proof of Thm. 4.1.* It remains to show that the map

$$\mathfrak{T}(\mathbf{x})(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}(s))\, \mathrm{d}s$$

is a contraction map in the space $C(J)$.

To this end we simply take a difference. Let $\mathbf{x}, \mathbf{y} \in C(J)$.

$$\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y}) = \int_{t_0}^{t} f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))\, \mathrm{d}s.$$

Since $f$ is Lipschitz in its second argument, for some constant $K$ (maybe even $K_s$),

$$\|\mathfrak{T}(\mathbf{x}) - \mathfrak{T}(\mathbf{y})\|_{C(J)} \leq \sup_{t\in J} \int_{t_0}^{t} |f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))|\, \mathrm{d}s \leq \eta \sup_{s\in J} K_s \|\mathbf{x} - \mathbf{y}\|_{C(J)},$$

by the triangle inequality.

Now we simply require $\eta$ to be small enough such that $\eta \sup_{s\in J} K_s < 1$. This will give us a contraction map, and by the previous lemma, a unique solution on $C(J)$ to the Cauchy problem in the theorem statement. In particular, the solution will be the limit under the $C(J)$ norm of the iterants in the following PICARD ITERATION:

$$\mathbf{x}_{n+1}(t) := \mathbf{x}(t_0) + \int_{t_0}^{t} f(s, \mathbf{x}_n(s))\, \mathrm{d}s.$$

□

*Remark* 4.1 (Maximal time of existence and bootstrapping). Suppose $\sup_{s\in\mathbb{R}} K_s$ is bounded. Then when we reach $t_0 + \eta$, we can extend it by another $\eta$, and then again, ad infinitum, and thus "bootstrap" our way to a globally unique solution. This would fail if the sequence of $\eta$ defined by

$$t_n := \sup J_n, \quad J_n := J_{n-1} \cup (t_{n-1} + \eta_n), \quad \eta_n \sup_{s\in J_n} K_s < 1$$

sums to a convergent series.

■

4.1. **Some words on the Peano existence theorem.** It turns out that it is possible to weaken the condition of Lipschitz continuity and establish an existence theorem with continuity only. This is known as Peano's existence theorem:

**Theorem 4.5.** *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. Then there exists a non-empty interval $[t_0, t_0 + \eta)$ on which Cauchy problem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(t, \mathbf{x}(t)), \qquad \mathbf{x}(t_0) = \mathbf{b}$$

*has a solution in $C([t_0, t_0 + \eta))$.*

But as we saw in Example 4.2, this solution may not be unique. Another way of looking at it is that an initial condition is not enough information to specify a unique solution in the space of continuous functions if $f$ is just continuous in both of its arguments.

## 5. Lecture V: Local Well-Posedness II

5.1. **Gronwall's Inequality.** Also called Gronwall's lemma, this inequality is the archetypal bound arising from a differential inequality:

**Theorem 5.1** (Gronwall's Inequality)**.** *Let $g : [0,T] \to \mathbb{R}$ be continuous and suppose that there are is a non-negative constant $C$ and a non-negative $v : [0,T] \to \mathbb{R}$ such that*

$$g(t) \leq C + \int_0^t v(s)g(s)\,\mathrm{d}s, \qquad t \in [0,T]. \tag{15}$$

*Then*

$$g(t) \leq C \exp\left(\int_0^t v(s)\,\mathrm{d}s\right).$$

*Remark* 5.1. Note that this is slightly more general than the inequality found in *Schaeffer and Cain*. One can understand the inequality (15) as a differential inequality if $C = g(0)$, and a differential formulation from which we can deduce the integral formulation (15) is

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) \leq v(t)g(t).$$

If $v$ is bounded on $[0,T]$, then we also have

$$g(t) \leq Ce^{\|v\|_{L^\infty([0,T])}t}, \qquad \left(\|v\|_{L^\infty([0,T])} := \sup_{t\in[0,T]} |v(t)|\right).$$

*Proof.* One simple way to prove the inequality is by iterating it and using the Taylor expansion for the exponential.

We can also set

$$G(t) = C + \int_0^t v(s)g(s)\,\mathrm{d}s,$$

from which we obtain

$$g(t) \leq G(t), \qquad G'(t) = v(t)g(t).$$

Leibnitz's rule then shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_0^t v(s)\,\mathrm{d}s} G(t)\right) = e^{-\int_0^t v(s)\,\mathrm{d}s} v(t)\big(-G(t) + g(t)\big) \leq 0,$$

because the exponential and $v$ are both non-negative, and $g \leq G$ pointwise.

Therefore we find that

$$e^{-\int_0^t v(s)\,\mathrm{d}s} G(t) \leq G(0) = C,$$

which can be expanded to yield the inequality in the theorem statement.

□

We shall apply Gronwall's inequality to derive continuous dependence on initial conditions.

**Corollary 5.2.** *Let $\mathbf{x}$ and $\mathbf{y}$ be solutions in $C([0,T])$ to the differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = f(t, \mathbf{u}(t)),$$

*with initial conditions $\mathbf{u}(0) = \mathbf{x}(0)$ and $\mathbf{u}(0) = \mathbf{y}(0)$, respectively. Suppose $f$ is continous $\mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and Lipschitz in its second argument:*

$$|f(s,\boldsymbol{\xi}) - f(s,\boldsymbol{\zeta})| \leq K_s|\boldsymbol{\xi} - \boldsymbol{\zeta}|, \qquad 0 \leq K_s \in C([0,T]), \quad \boldsymbol{\xi},\boldsymbol{\zeta} \in \mathbb{R}^d.$$

*For $t \in [0,T]$, it holds that*

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq |\mathbf{x}(0) - \mathbf{y}(0)|e^{\int_0^t K_s\,\mathrm{d}s}.$$

*Proof.* This result follows from the previous theorem by application of the Lipschitz assumption of $f$. First integrate the differential equation over $[0,t]$, $t \in [0,T]$ with initial conditions $\mathbf{x}(0)$ and $\mathbf{y}(0)$, then take the difference. By the triangle inequality,

$$\begin{aligned}|\mathbf{x}(t) - \mathbf{y}(t)| &\leq |\mathbf{x}(0) - \mathbf{y}(0)| + \int_0^t |f(s, \mathbf{x}(s)) - f(s, \mathbf{y}(s))| \, \mathrm{d}s \\ &\leq |\mathbf{x}(0) - \mathbf{y}(0)| + \int_0^t K_s |\mathbf{x}(s) - \mathbf{y}(s)| \, \mathrm{d}s.\end{aligned}$$

Now we can apply the Gronwall inequality with $g(t) = |\mathbf{x}(t) - \mathbf{y}(t)|$ and $v(s) = K_s$ in (15). □

## 7. Lecture VII: Hyperbolic Critical Points

Starting this lecture and for the subsequent few lectures, we shall be looking at what may be known as the local theory of nonlinear systems. The word "local" here is used in a sense different to "local" well-posedness. By "local" here we mean "local in the phase space" and not "local in time".

### 7.1. **Linearization.**

Having deduced the existence and uniqueness of a solution to the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \tilde{f}(t, \mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{b}$$

where $\tilde{f}$ is Lipschitz in $\mathbf{x}$ and continuous in $t$, and explored questions of continuous dependence, we shall now look again at asymptotic behaviour of solutions. To do so we shall have to focus on $\tilde{f}$ for which $\tilde{f}$ has a bounded Lipschitz constant over all of $\mathbb{R}^d$ and not just on an open set $U \subseteq \mathbb{R}^d$ around the initial value. We shall also restrict our attention to autonomous systems again.

Looking at asymptotic behaviours compels us to consider again the behaviour of the set of fixed, or equilibrium points, of the system. That is, recalling the three possible behaviour as $t \to \infty$ — that of escape to infinity (along, perhaps, a particular direction), limit at a point, or limit in a more general set (such as around a periodic orbit), we shall first discuss the second case. A FIXED, or CRITICAL, or EQUILIBRIUM POINT of the dynamics is a point $\mathbf{x}_0 \in \mathbb{R}^d$ for which

$$\tilde{f}(\mathbf{x}_0) = 0.$$

Starting from such a point, we see that $\mathrm{d}\mathbf{x}/\mathrm{d}t = 0$, so that $\mathbf{x}(t) = \mathbf{x}_0$ for all time. This point is then a FIXED POINT of the flow because $\phi_t(\mathbf{x}_0) = \mathbf{x}_0$ for every $t$. At each coordinate, $\mathbf{x}_0$ solves $\tilde{f}^i(\mathbf{x}) = 0$. The codimension one surface $\tilde{f}^i(\mathbf{x}) = 0$ is known as a NULLCLINE, and the fixed point is at the intersection of all the nullclines.

By Taylor's theorem, where $f$ is continuously differentiable around $\mathbf{x}_0$ up to all orders, we can expand $f$ about $\mathbf{x}_0$ as

$$\tilde{f}(\mathbf{x}) = \mathrm{D}\tilde{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}\mathrm{D}^2\tilde{f}(\mathbf{x}_0) : (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) + \dots.$$

The derivative $\mathrm{D}\tilde{f}(\mathbf{x}_0)$ is the matrix with entries

$$(\mathrm{D}\tilde{f}\big|_p)^j_i = \left.\frac{\partial \tilde{f}^j}{\partial x^i}\right|_{\mathbf{x}_0}.$$

Therefore one can make the approximation

$$\tilde{f}(\mathbf{x}) \approx \mathrm{D}\tilde{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

in a neighbourhood of $\mathbf{x}_0$ the sense of the usual vector norm in $\mathbb{R}^d$, the remaining terms being of order $O(|\mathbf{x} - \mathbf{x}_0|^2)$.

It turns out that the *dynamics* of the *linear* system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = A(\mathbf{x} - \mathbf{x}_0), \qquad A = D\tilde{f}(\mathbf{x}_0)$$

also approximates the dynamics of the full nonlinear system in a neighbourhood of $\mathbf{x}_0$ in a sense to be made precise later. We shall also see that this is not as absolute an approximation as pointwise as vectors, because this approximation-in-dynamics can fail spectacularly when $A$ does not generate a hyperbolic flow. (Recall from §3.2 that a flow is hyperbolic when $A$ does not have eigenvalues with zero real parts.)

To this end we shall begin by restricting our attention even more narrowly and consider autonomous first order systems around critical points $\mathbf{x}_0$ at which the eigenvalues of $\mathrm{D}\tilde{f}(\mathbf{x}_0)$ all have non-zero real parts. We shall call these critical points HYPERBOLIC CRITICAL POINTS. Our first

task will be to make a small linear adjustment to our problem, by shifting everything by $\mathbf{x}_0$ so that the critical point of interest is the origin:

Let $f(\mathbf{x}) = \tilde{f}(\mathbf{x} + \mathbf{x}_0)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

will have a critical point at $\mathbf{0} \in \mathbb{R}^d$ if, and only if, $\tilde{f}(\mathbf{x}_0) = 0$.

7.2. **Examples.** In the following we shall consider a few examples of hyperbolic critical points. These examples are chosen because we shall revisit them. They are all systems with dependence on at least one parameter other than time.

**Example 7.1** (Van der Pol's Equation)**.** The first example we inspect is known as van der Pol's Equation/Oscillator, which models oscillatory currents in electrical circuits with vacuum tubes — essentially and oscillator with non-linear damping. For $\beta \in \mathbb{R}$, the Van der Pol Equation is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\beta(x^2 - 1)y - x. \end{aligned} \tag{16}$$

This equation arose in is often stated in another form. We can find its fixed points relatively easily. In fact it only has one:

$$0 = y, \qquad 0 = -\beta(x^2 - 1)y - x \qquad \implies x = y = 0.$$

We can find $\mathrm{D}f$:

$$\mathrm{D}f = \begin{pmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ \partial f^2/\partial x & \partial f^2/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\beta xy - 1 & \beta \end{pmatrix}.$$

Therefore

$$\mathrm{D}f\Big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \beta \end{pmatrix},$$

and $\mathrm{D}f|_{(0,0)}$ is non-singular for $\beta \neq 0$.

Its eigenvalues are

$$\lambda_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}.$$

This means that the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f|_{(0,0)}\mathbf{x}(t)$$

has an unstable focus when $0 < \beta < 2$, and an unstable node when $2 < \beta$. Either way it is a SOURCE when $\beta > 0$.

Because of the physical system which it models one does not often consider the case $\beta < 0$. But we are free to do so here: When $-2 < \beta < 0$, the linearized system has a stable focus and when $\beta < -2$, the linearized system has a stable node. That is, the system is a SINK when $\beta < 0$.

Therefore we should expect that the dynamics of the nonlinear system exhibit the respective behaviours in these ranges of $\beta$ in a small region around $(x, y) = (0, 0)$.

**Example 7.2** (Duffing's Equation)**.** The Duffing Equation/Oscillator models damped oscillators:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^3 - \beta y. \end{aligned} \tag{17}$$

Here $\beta > 0$ is the damping parameter. This equation is usually written with an inhomogeneity that also models a driving force. But without such a forcing term, we have an autonomous system whose fixed points are at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ x - x^3 - \beta y \end{pmatrix} \qquad \implies \qquad (x, y) \in \{(0, 0), (\pm 1, 0)\}.$$

The first order approximation is

$$\mathrm{D}f = \begin{pmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ \partial f^2/\partial x & \partial f^2/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & -\beta \end{pmatrix}$$

At the fixed point $(x, y) = (0, 0)$,

$$\mathrm{D}f|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -\beta \end{pmatrix}.$$

and its eigenvalues are

$$\lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2+4}}{2}.$$

This implies that the linearized system has a saddle for all $\beta > 0$.

At the fixed points $(x, y) = (\pm 1, 0)$,

$$\mathrm{D}f|_{(\pm 1,0)} = \begin{pmatrix} 0 & 1 \\ -2 & -\beta \end{pmatrix}.$$

and its eigenvalues are

$$\lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2-8}}{2}.$$

This is an SINK for all $\beta > 0$, and so the linearized system exhibits a stable node when $\beta > \sqrt{8}$, and a stable focus when $\beta < \sqrt{8}$.

Stability is what we expect of damping, and we expect that the dynamics of the nonlinear system exhibit the respective behaviours in these ranges of $\beta$ in a small region around each fixed point.

**Example 7.3** (Lotka-Volterra model)**.** The second example we consider ithe Lotka-Volterra model. This equation models predator-prey populations dynamics:

$$\begin{aligned} \dot{x} &= x - xy \\ \dot{y} &= \rho(xy - y). \end{aligned} \tag{18}$$

The prey concentration is modelled by $x$, and the predator concentration is modelled by $y$. Interaction/predation is modelled by $xy$. The parameter $\rho$ is positive.



First we seek the critical points:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x - xy \\ \rho(xy - y) \end{pmatrix} \implies x = y = \pm 1 \quad \text{or} \quad x = y = 0.$$

Again it is simple to find the linearization here:

$$\mathrm{D}f = \begin{pmatrix} 1-y & -x \\ \rho y & \rho(x-1) \end{pmatrix}.$$

At the fixed points, we find:

$$\mathrm{D}f|_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix}, \qquad \mathrm{D}f|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}, \qquad \mathrm{D}f|_{(-1,-1)} = \begin{pmatrix} 2 & 1 \\ -\rho & 2\rho \end{pmatrix}.$$

The eigenvalues of $\mathrm{D}f$ at $(1, 1)$ are $\lambda_\pm = \pm i\sqrt{\rho}$. Therefore $(1, 1)$ is not a hyperbolic critical point.

At $(0, 0)$, the eigenvalues of $\mathrm{D}f$ are $\lambda_1 = 1$ and $\lambda_2 = -\rho$. Eigenvectors $(0, 1)^\top$ and $(1, 0)^\top$, respectively, are also more readily found than in the previous two examples. Therefore the behaviour of the linearized is that of a saddle for all $\rho > 0$, and we expect the behaviour of the nonlinear system to be the same around the point $(0, 0)$.

It makes little sense to study the equation at $(x, y) = (-1, -1)$ if we consider the model, but we are not thereby hindered. The eigenvalues of $\mathrm{D}f|_{(-1,-1)}$ are $\lambda_\pm = 1 + \rho \pm \sqrt{\rho^2 - 3\rho + 1}$. Therefore according as $\rho \in (3 - \sqrt{5}, 3 + \sqrt{5})$ or $\rho \in \mathbb{R}_{>0} \backslash [3 - \sqrt{5}, 3 + \sqrt{5}]$, the linearized system is an unstable focus or node. And we expect the nonlinear system to behave likewise in a small region around $(-1, -1)$ when $\rho$ are in these intervals.

**Example 7.4** (Augmented Lotka-Volterra model)**.** The augmented Lotka-Volterra model amends some deficiencies in the Lotka-Volterra model that we can glean from our analysis in the previous example.

Since the model is a saddle around $(0,0)$, (with eigenvectors $(0,1)^\top$ and $(1,0)^\top$, respectively) we see that no matter how small $x$ or $y$ gets, as long as they are positive, extinction does not happen.

Secondly, without predators ($y(0)=0$), the prey population/concentration can grow indefinitely.

We introduce two parameters — a self-sustaining population parameter $0<\varepsilon<1$ and a carrying capacity parameter $K>\varepsilon$ — to rectify these two problems in the modified equation:

$$\begin{aligned}\dot{x} &= x\left(\frac{x-\varepsilon}{x+\varepsilon}\right)\left(1-\frac{x}{K}\right)-xy\\ \dot{y} &= \rho(xy-y).\end{aligned} \tag{19}$$

Let us write $\phi$ for

$$\frac{x-\varepsilon}{x+\varepsilon}\left(1-\frac{x}{K}\right).$$

The critical points in $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ are at

$$x\phi(x)=xy, \qquad xy=y \qquad \Longrightarrow \qquad (x,y)\in\{(0,0),(\varepsilon,0),(K,0),(1,\phi(1))\}.$$

The gradient is

$$\mathrm{D}f=\begin{pmatrix} x\phi'(x)+\phi(x)-y & -y\\ \rho y & \rho(x-1)\end{pmatrix}.$$

and

$$\phi'(x)=\frac{-1}{K}\frac{x-\varepsilon}{x+\varepsilon}+\frac{2\varepsilon}{(x+\varepsilon)^2}\left(1-\frac{x}{K}\right).$$

At $(x,y)=(0,0)$, both species are extinct — this is the extinction equilibrium. The eigenvalues here are $\lambda_1=-1$ and $\lambda_2=-\rho$. This implies that the linearized dynamics has a stable node, and the nonlinear dynamics has one in a vicinity of $(x,y)=(0,0)$.

At $(x,y)=(\varepsilon,0)$ the predators are absent and the prey are at extinction threshold. The eigenvalues here are $\lambda_1=(1-\varepsilon/K)/2$ and $\lambda_2=-\rho(1-\varepsilon)$. This implies that the linearized dynamics has a saddle, and the nonlinear dynamics has one in a vicinity of $(x,y)=(\varepsilon,0)$.

At $(x,y)=(K,0)$ the predators are absent and the prey are at a healthy concentration. The eigenvalues here are $\lambda_1=-(K-\varepsilon)/(K+\varepsilon)$ and $\lambda_2=\rho(K-1)$. This implies that the linearized dynamics has a stable node or a saddle, according as $K<1$ or $K>1$, and the nonlinear dynamics has one in a vicinity of $(x,y)=(K,0)$.

At $(x,y)=(1,\phi(1))$ the predators are conextant — this is the coexistence equilibrium. The eigenvalues here are

$$\lambda_\pm=\frac{\phi'(1)\pm\sqrt{(\phi'(1))^2-4\rho\phi^2(1)}}{2}, \qquad \phi'(1)=\frac{2\varepsilon-(1+2\varepsilon-\varepsilon^2)/K}{(1+\varepsilon)^2},$$

which can still be hyperbolic, and is a sink or source according as $2\varepsilon-(1+2\varepsilon-\varepsilon^2)/K$ is negative or positive. It is a furthermore node or focus depending on whether $\phi'(1))^2-4\rho\phi^2(1)$ is positive or negative. The nonlinear dynamics has corresponding behaviour in a vicinity of $(x,y)=(1,\phi(1))$.

**Example 7.5** (Activator-Inhibitor systems)**.** The activator-inhibitor models the concentrations of two species of chemicals, the one modelled by $x$ is produced more rapidly as its concentration increases, and is thereby self-promoting, and the one modelled by $y$ inhibits the production of $x$. This model is given by the equation:

$$\begin{aligned} \dot{x} &= \sigma \frac{x^2}{1+y} - x \\ \dot{y} &= \rho(x^2 - y). \end{aligned} \tag{20}$$

The parameters $\sigma$ and $\rho$ are positive.

Again we can find the fixed points, and they are:

$$(x, y) \in \{(0,0), (r_+, r_+^2), (r_-, r_-^2)\},$$

where

$$y = \sigma x - 1, \quad y = x^2 \qquad \Longrightarrow \qquad r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

Therefore the fixed points not at zero only exist in the phase space when $\sigma > 2$.

The linearized system is determined by

$$\mathrm{D}f = \begin{pmatrix} 2\sigma x/(1+y) - 1 & -\sigma x^2/(1+y)^2 \\ 2\rho x & -\rho \end{pmatrix}.$$

At $(x, y) = (0, 0)$, we see that $\mathrm{D}f|_{(0,0)}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \rho$. This implies a saddle for the linearized dynamics and a saddle for the nonlinear dynamics near $(0, 0)$.

Suppose $\sigma > 2$. Notice that on the nullcline $y = \sigma x - 1$,

$$\mathrm{D}f\Big|_{y=\sigma x-1} = \begin{pmatrix} 1 & -1/\sigma \\ 2\rho x & -\rho \end{pmatrix}.$$

That means

$$\mathrm{D}f\Big|_{(r_\pm, r_\pm^2)} = \begin{pmatrix} 1 & -1/\sigma \\ 2\rho r_\pm & -\rho \end{pmatrix}.$$

This has eigenvalues

$$\lambda_1^\pm = \frac{(1-\rho) + \sqrt{(1-\rho)^2 \mp 4\rho\sqrt{1 - 4/\sigma^2}}}{2}, \qquad \lambda_2^\pm = \frac{(1-\rho) - \sqrt{(1-\rho)^2 \mp 4\rho\sqrt{1 - 4/\sigma^2}}}{2}.$$

Therefore the behaviour at $(x, y) = (r_-, r_-^2)$ is approximated by a saddle. The behaviour at $(x, y) = (r_+, r_+^2)$ is a source or a sink according as $\rho < 1$ or $\rho > 1$.

## 8. Lecture VIII: Embedded Submanifolds of $\mathbb{R}^d$

We shall now delve deeper into the qualitative local theory of autonomous ODEs and understand in exactly what way the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{x}_0)\mathbf{x}$$

approximates the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

near a hyperbolic critical point. This discussion will be quite geometrical and we shall acquire some language before the undertaking.

8.1. **Manifolds.** In this subsection we shall be discussing some tools that shall allow us to talk about the dynamics of one system approximating the dynamics of another system, as well as allow us to write down a structure theorem for nonlinear autonomous dynamical systems around critical points, similar to Theorem 3.3 for linear systems. This subsection is not per se examinable.

Without digressing inordinately to define and familarize ourselves with the notion of topology and a topological space in general, let us simply say that one can think of a "topology" as the structures on a set that characterize, or are characterized by, how convergence takes place. We are generally only concerned with $\mathbb{R}^d$ in this module, and when we need to consider convergence on other spaces (such as $C(J)$ or the space of matrices (which is essentially $\mathbb{R}^{d\times d}$)), we shall make a digression for that particular space.

The important thing to notice here is that if $f : \mathbb{R}^d \to \mathbb{R}^n$ is a continuous function, then that continuity is connected with how sequences converge in one space as compared to how they converge in the other space. Suppose $\mathbf{x}_n \to \mathbf{x}$ is a converging sequence in $\mathbb{R}^d$. Then the continuity of $f$ implies the convergence of $\mathbf{y}_n = f(\mathbf{x}_n)$ to $\mathbf{y} = f(\mathbf{x})$. Conversely, the continuity of $f$ is implied by the property that it transfers convergence from one space to the other. After all, continuity means that if the pre-image changes a little, then the image also changes just a little. Topology lets us talk about "nearness" on a space.



*Proof.* If $\mathbf{x}_n \to \mathbf{x}$, then for every $\delta > 0$, there is an $N$ such that $n > N$ implies $|\mathbf{x}_n - \mathbf{x}| < \delta$.

If $f$ is continuous, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\mathbf{x}-\mathbf{y}| < \delta$ implies $|f(\mathbf{x})-f(\mathbf{y})| < \varepsilon$.

Therefore, fixing $\varepsilon > 0$, we can find a large enough $N$ such that $|\mathbf{x}_n-\mathbf{x}| < \delta$, and $|f(\mathbf{x}_n)-f(\mathbf{x})| < \varepsilon$. This implies that $f(\mathbf{x}_n) \to f(\mathbf{x})$. □

Therefore, the structures that allow for convergence — the topology — can be characterized thus: we say that two topological spaces $X$ and $Y$ are topologically the same, or HOMEOMORPHIC, if there is a continuous map $H : X \to Y$ that is invertible, and its inverse $H^{-1} : Y \to X$ is also continuous. Such a map is called a HOMEOMORPHISM. On $\mathbb{R}^d$ we can put more structures on $H$. A $C^k$-DIFFEOMORPHISM is a $k$-times continuously differentiable map with an inverse that is also $k$-times continuously differentiable. If $k$ is unbounded, we say that the diffeomorphism is SMOOTH.

On $\mathbb{R}^d$, $\mathbf{x}_n \to \mathbf{x}$ if on any open ball $B_\delta(\mathbf{x})$, one finds that $\mathbf{x}_n \in B_\delta(\mathbf{x})$ if $n > N$ for some $N = N(\delta)$. Therefore, the collection of open balls on $\mathbb{R}^d$ is the structure that controls convergence, or nearness. All $\mathbf{x}_n$ with $n > N$ are $\delta$-near to $\mathbf{x}$. An arbitrary union of open balls is an open set, and not unreasonably an open set containing $\mathbf{x}$ is known as a NEIGHBOURHOOD of $\mathbf{x}$.

Heuristically speaking, an $m$-dimensional manifold is a topological space $M$ that *looks locally like* $\mathbb{R}^m$. That is, around each point $p \in M$, there is a neighbourhood $U \subseteq M$ and a homeomorphism $\varphi_U : U \to V \subseteq \mathbb{R}^m$ ($\varphi$ is continuous and invertible and its inverse is continuous), where $V$ is a neighbourhood in $\mathbb{R}^m$. Homeomorphisms defined on overlapping neighbourhoods are related in ways that encode how $M$ is patched together from little pieces of $\mathbb{R}^d$, much like a globe is patched together with flat strips and two caps.

In this module we shall not be needing an intrinsic definition of a manifold. Insofar as we are concerned, we shall use the word "manifold" to mean a level set of a continuous function $h : \mathbb{R}^d \to \mathbb{R}^n$ because all our manifolds are naturally embedded in phase space, which is $\mathbb{R}^d$. That is, by MANIFOLD, we shall mean a subset $X \subseteq \mathbb{R}^d$ defined by a $C^1$ function $h : \mathbb{R}^d \to \mathbb{R}^n$ of constant rank $k$, and a constant $\mathbf{c} \in \mathbb{R}^n$ thus:

$$X = \{\mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{c}\},$$

and inheriting the metric properties of $\mathbb{R}^d$ as a subspace. The RANK of a function $h : \mathbb{R}^d \to \mathbb{R}^n$ at the point $p \in \mathbb{R}^d$ is the the dimension of the column space of its gradient, $\mathrm{D}h$, which in geometry is also variously denoted $\nabla h$, grad$h$, d$h$, or $h_*$.

By the zero map $\mathbf{Z} : \mathbb{R}^d \to \mathbb{R}$, given by $\mathbf{Z}(\mathbf{x}) = 0$, it is immediate that $\mathbb{R}^d$ itself is a manifold — as it must be even by the more general heuristic definition of a manifold earlier given — $\mathbb{R}^d$ looks (globally and) locally like $\mathbb{R}^d$.

To get a quick idea of what $X$ must look like and why the constant rank condition is important, we need look no further than the rank theorem:

**Theorem 8.1.** *Let $M$ and $N$ be $C^1$-manifolds of dimensions $m$ and $n$, respectively, and let $F : M \to N$ be a $C^1$-map of constant rank $k < m$. For each $p \in M$, there exist $C^1$-diffeomorphisms $\varphi : M \to M$ defined on a neighbourhood of $p$ and $\psi : N \to N$ defined on a neighbourhood of $F(p)$ such that*

$$\psi \circ F(\varphi^{-1}(p)) = ((\varphi^{-1})^1(p), (\varphi^{-1})^2(p), \ldots, (\varphi^{-1})^k(p), 0, \ldots, 0).$$

From this theorem it is immediate that

**Corollary 8.2.** *If $h : \mathbb{R}^d \to \mathbb{R}^n$ is a $C^1$-map of constant rank $k$, then each level set of $h$ is a manifold of codimension $k$.*

The rank theorem is a ready consequence of the the Inverse Function Theorem below. These theorems will then also be useful for helping us understand how to associate linearized dynamics with the full dynamics of a nonlinear system.



**Theorem 8.3** (Inverse Function Theorem)**.** *Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $F : U \to V$ be a $C^1$-map. If $\mathrm{D}F(p)$ is non-singular at some point $p \in U$ then there are connected neighbourhoods $U_0 \subseteq U$ of $p$ and $V_0 \subseteq V$ of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a $C^1$-diffeomorphism.*

Whlist we shall not prove this theorem here, let us mention that it can be deduced with the aid of the contraction mapping principle. Related to the Inverse Function Theorem is the Implicit Function Theorem:

**Theorem 8.4** (Implicit Function Theorem)**.** *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open set, let $(\mathbf{x}, \mathbf{y}) = (x^1, \ldots, x^n, y^1, \ldots y^k)$ be the standard coordinates on $U$. Suppose $\Phi : U \to \mathbb{R}^k$ is a $C^1$-map, $(\mathbf{a}, \mathbf{b}) \in U$, and $\Phi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$. If the matrix $\mathrm{D}\Phi(\mathbf{a}, \mathbf{b})$ is non-singular, then there exist neighbourhoods $V_0 \subseteq \mathbb{R}^n$ of $\mathbf{a}$ and $W_0 \subseteq \mathbb{R}^k$ of $\mathbf{b}$ and a smooth map $F : V_0 \to W_0$ such that*

$$\Phi(\mathbf{x}, \mathbf{y}) = \mathbf{c} \quad \leftrightarrow \quad \mathbf{y} = F(\mathbf{x}).$$

Finally we need to introduce the tangent space. Let $M$ be a manifold of dimension $n$ defined by $\{\mathbf{x} \in \mathbb{R}^d : h(\mathbf{x}) = \mathbf{c}\}$, where $h : \mathbb{R}^d \to \mathbb{R}^{d-n}$ is a $C^1$ map of constant rank. Let $p \in M$. The TANGENT SPACE of $M$ at $p$ is denoted by $T_pM$. Naturally all the vectors that can be imagined as having one end at $p$ and "lying in $M$" would be directions along which $h$ does not change, because $M$ is by definition a level set. Therefore, we can identify the tangent space $T_pM$ with

$$T_pM = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} - p \in \ker \mathrm{D}h\}.$$

This is reduces to the usual definition of a tangent space if we take $M$ to be of codimension one, or in fact, a tangent plane if $d = 3$, and $n = 2$. Since $\mathrm{D}h$ is zero in the $n$ directions of $M$ and nonzero in the remaining $d - n$ dimensions, we see that $T_pM$ is in fact isomorphic to the vector space $\mathbb{R}^n$.

## 9. Lecture IX: Stable Manifold and Hartman-Grobman Theorems

9.1. **Stable Manifold Theorem.** Now we are ready to discuss the ideas of the stable manifold theorem.

Recall that for a linear system, by Thm.3.3, we can decompose phase space about its critical points to $\mathbb{R} = E^s \oplus E^c \oplus E^u$, where

$$E^s = \text{span} \bigcup_{\{n:\Re\lambda_n<0\}} V_n$$
$$E^c = \text{span} \bigcup_{\{n:\Re\lambda_n=0\}} V_n$$
$$E^u = \text{span} \bigcup_{\{n:\Re\lambda_n>0\}} V_n,$$

and $V_n$ were the vectors spanning the general eigenspace associated with $\lambda_n$. Now if $\mathbf{0}$ is a hyperbolic critical point, by definition, $\mathrm{D}f(\mathbf{0})$ does not have eigenvalues with $\Re\lambda_n = 0$. Therefore, for the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{0})\mathbf{x}(t),$$

we have a decomposition of phase space into $\mathbb{R}^d = E^s \oplus E^u$.

The stable manifold theorem is as follows:

**Theorem 9.1** (Stable Manifold Theorem)**.** *Let $U$ be an open subset of $\mathbb{R}^d$ containing the origin. Let $f \in C^1(U;\mathbb{R}^d)$, and let $\phi_t$ be the flow of the nonlinear system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x}).$$

*Suppose that $f(\mathbf{0}) = \mathbf{0}$ and $\mathrm{D}f(\mathbf{0})$ has $k$ eigenvalues (counting multiplicity) with negative real parts and $d-k$ eigenvalues with positive real parts. Then*

(i) *there exists a dimension $k$ $C^1$-manifold $M_s$ tangent to $E^s$ of the linearized system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathrm{D}f(\mathbf{0})\mathbf{x}$$

*at $\mathbf{0}$ such that for all $t \geq 0$, $\phi_t(M_s) \subseteq M_s$ and for all $\mathbf{y} \in M_s$,*

$$\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0};$$

*and*

(ii) *there exists a dimension $d-k$ $C^1$-manifold $M_u$ tangent to $E^u$ of the linearized system at $\mathbf{0}$ such that for all $t \leq 0$, $\phi_t(M_u) \subseteq M_u$ and for all $\mathbf{y} \in M_u$,*

$$\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}.$$

**Example 9.1.** We look at a nonlinear system that we can solve explicitly:

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2 + x_1^2$$
$$\dot{x}_3 = x_3 + x_1^2.$$

There is one fixed point, which is the origin. The linearization is given by

$$\mathrm{D}f(\mathbf{0}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and so $\mathbf{0}$ is a hyperbolic critical point. The eigenvalues and eigenvectors are readily deducible, and we see that

$$E^s = \{\mathbf{x} : x_2 = 0\} = \text{span}\{(1,0,0)^\top, (0,1,0)^\top\}, \qquad E^u = \{\mathbf{x} : x_1 = x_2 = 0\} = \text{span}\{(0,0,1)^\top\}.$$

The equations can also be integrated by hand, giving

$$\begin{aligned} x_1(t) &= y_1 e^{-t} \\ x_2(t) &= y_2 e^{-t} + y_1^2(e^{-t} - e^{-2t}) \\ x_3(t) &= y_3 e^{t} + y_1^2(e^{t} - e^{-2t})/3, \end{aligned}$$

with $\mathbf{x}(0) = \mathbf{y} = (y_1, y_2, y_3)^\top$.

We see that $\lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1^2/3 + y_3 = 0$, and so

$$M_s = \{\mathbf{y} \in \mathbb{R}^3 : y_1^2 + 3y_3 = 0\}.$$

Likewise, $\lim_{t\to-\infty} \phi_t(\mathbf{y}) = \mathbf{0}$ if, and only if, $y_1 = y_2 = 0$, and so

$$M_u = \{\mathbf{y} \in \mathbb{R}^3 : y_1 = y_2 = 0\}.$$

It is clear that $M_u$ is tangent to $E^u$ at $\mathbf{0}$ because they coincide entirely. Taking the derivative of $h(y_1, y_2, y_3) = y_1^2 + 3y_3$, for which $M_s$ is the level set at 0, we find

$$\nabla h\Big|_{\mathbf{0}} = (2y_1, 0, 3)^\top|_{(0,0,0)} = (0,0,3),$$

which is indeed perpendicular to $E^s$, and so $S$ and $E^s$ are tangent at $\mathbf{0}$, as expected.



The way that the Stable Manifold Theorem is usually proven gives us insight into the structure of nonlinear systems. And whilst we shall not be proving the Stable Manifold Theorem, it is of benefit to discuss some elements of its proof. First notice that for a general first order, autonomous nonlinear system, we have the following Taylor's expansion around a hyperbolic critical point $\mathbf{x}_0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\mathbf{x}_0)\mathbf{x}(t) + \mathbf{G}(\mathbf{x}),$$

where $\mathbf{G}$ has zero first derivative at $\mathbf{x}_0$. This means that whilst $\mathbf{G}$ might not be "second-order" in $\mathbf{x}$, for every $\varepsilon$, there is a $\delta$ such that if $|\mathbf{x} - \mathbf{x}_0| < \delta$, and $|\mathbf{y} - \mathbf{x}_0| < \delta$,

$$|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})| \leq \varepsilon|\mathbf{x} - \mathbf{y}|.$$

By applying the Jordan Normal Form Theorem (Thm. 3.1), we can assume that $\mathrm{D}f(\mathbf{x}_0)$ is of the form

$$\mathrm{D}f(\mathbf{x}_0) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where $P$ is a matrix in Jordan normal form with only eigenvalues of negative real parts, and $Q$ is a matrix in Jordan normal form with only eigenvalues of positive real parts. The linear system can be solved by exponentiating $\mathrm{D}f(\mathbf{x}_0)$ so that where

$$W(t) = \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix},$$

the flow of the linearized system is

$$e^{\mathrm{D}f(\mathbf{x}_0)t} = W(t) + Z(t).$$

Using the Duhamel representation to treat the term $\mathbf{G}$ as an inhomogeneity, we find

$$\mathbf{x}(t) = e^{\mathrm{D}f(\mathbf{x}_0)t}\mathbf{x}(0) + \int_0^t e^{\mathrm{D}f(\mathbf{x}_0)(t-s)}G(\mathbf{x}(s))\,\mathrm{d}s$$

If we look at solutions that start on what might potentially be the stable manifold, we have solutions of the form

$$\mathbf{x}(t) = W(t)\mathbf{x}(0) + \int_0^t W(t-s)G(\mathbf{x}(s))\,\mathrm{d}s - \int_t^\infty Z(t-s)G(\mathbf{x}(s))\,\mathrm{d}s.$$

Now we can take Picard iterations and exploit the signs of the real parts of the eigenvalues to bound $W$ and $Z$ to reach our conclusions. A similar procedure can be done for the unstable manifold, except we then run time in the backwards direction using the reversal $t \mapsto -t$.

The Stable Manifold Theorem only defines $M_s$ and $M_u$ on a small neighbourhood of the hyperbolic critical point. To supplement their definition in the theorem we also introduce the GLOBAL STABLE AND UNSTABLE MANIFOLDS at $\mathbf{0}$ if it is a hyperbolic fixed point:

$$W^s(\mathbf{0}) = \bigcup_{t\le 0} \phi_t(M_s)$$
$$W^u(\mathbf{0}) = \bigcup_{t\ge 0} \phi_t(M_u).$$

These may not be manifolds in the sense we have defined, or in the more general sense conventionally used, except restricted to a neighbourhod of the hyperbolic critical point, but they are flow invariant, and satisfy the properties respectively ascribed to $M_s$ and $M_u$ in the Stable Manifold Theoremm. This is primarily because the function of which they are level sets can fail to be constant rank, and the "manifold" can intersect itself, so when we say "$C^k$-manifold" below, we mean essentially that it is $C^k$ on neighbourhoods where the function defining it has the same rank.

We are also in a position to speak briefly of non-hyperbolic critical points:

**Theorem 9.2** (Centre Manifold Theorem)**.** *Let $f \in C^1(U;\mathbb{R}^d)$ and $f(\mathbf{0}) = \mathbf{0}$. Suppose $\mathrm{D}f(\mathbf{0})$ has $k$ eigenvalues with negative real parts, $m$ eigenvalues with zero real parts, and $(d-k-m)$ eigenvalues with positive real parts. There exists*

(i) *an $m$-dimensional $C^1$-*CENTRE MANIFOLD *$W^c(\mathbf{0})$ tangent to the centre subspace $E^c$ of the linearized system at $\mathbf{0}$,*
(ii) *a $k$-dimensional $C^1$ stable manifold $W^s(\mathbf{0})$ tangent to the stable subspace $E^s$ of the linearized system at $\mathbf{0}$, and*
(iii) *a $(d-k-m)$-dimensional $C^1$ unstable manifold $W^u(\mathbf{0})$ tangent to the unstable subspace $E^u$ of the linearized system at $\mathbf{0}$.*

*These three subsets of $\mathbb{R}^d$ are invariant under the flow $\phi_t$.*



What happens on the centre manifold shall remain a mystery to us as long as we are only willing to look at approximations to first order because of another topological fact, this time of the real numbers. If $\lambda = \sigma + i\tau$, and $\sigma \neq 0$, then there is always a small enough perturbation of $\lambda$ by $h \in \mathbb{C}$ such that the sign of $\Re(\lambda+h)$ is the same as the sign of $\sigma$. Not being zero is an open condition. But if $\sigma = 0$, any (general) perturbation of $\lambda$ will give $\sigma$ a sign. Therefore, we see that what determines the behaviour on the centre manifold is determined by how the nonlinear terms perturb the system *spectrally* in a neighbourhood of a critical point. We shall find that at nonhyperbolic critical points, completely novel behaviours can arise because of nonlinearity.

9.2. **Hartman-Grobman Theorem.** First we seek to exhaust the facilities of first order approximative methods as well as we can with the tools available to us.

The Hartman-Grobman Theorem is a partial generalization of the Fundamental Theorem of Linear Systems, and gives us more precise information about trajectories near hyperbolic critical points. Recall that having defined the exponential of a matrix, the Fundamental Theorem allowed us to write a solution to the autonomous linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{b}$$

as

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{b}.$$

The Hartman-Grobman theorem allows us to write the solution to the nonlinear system near a hyperbolic critical point $\mathbf{x}_0$ as an approximation of

$$\exp(\mathrm{D}f(\mathbf{x}_0)t)\mathbf{b},$$

if the initial condition $\mathbf{x}(0) = \mathbf{b}$ is sufficiently close to $\mathbf{x}_0$.

To understand the contents of the precise statement, we define two terms. Suppose we have two first order autnomous systems in $\mathbb{R}^d$, shifted so that one of their critical points each are at $\mathbf{0}$. Let one system have a flow $\phi_t$ and the other have a flow $\psi_t$. These two autonomous first order systems are said to be TOPOLOGICALLY EQUIVALENT in a neighbourhood of $\mathbf{0}$ if there are two open sets $U$ and $V$, both containing $\mathbf{0}$, a homeomorphism $H : U \to V$ and a monotonically increasing function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\psi_{\eta(t)} \circ H(\mathbf{x}) = H \circ \phi_t(\mathbf{x}).$$

If we can choose $\eta$ (by changing $H$, say) so that $\eta = \mathrm{Id}_{\mathbb{R}}$, then the systems are said to be TOPOLOGICALLY CONJUGATE in a neighbourhood of $\mathbf{0}$.

**Theorem 9.3** (Hartman-Grobman Theorem)**.** *Let $\mathbf{x}_0 \in \mathbb{R}^d$ be a hyperbolic critical point of the system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t)),$$

*with flow $\phi_t$. There exists neighbourhoods $U$ and $V$ of $\mathbf{x}_0$, a homeomorphism $H : U \to V$ and an interval $I \in \mathbb{R}$ containing $0$ such that for every $\mathbf{y} \in U$ and $t \in I$,*

$$H \circ \phi_t(\mathbf{y}) = \exp(\mathrm{D}f(\mathbf{x}_0)t)H(\mathbf{y}).$$

Hartman also showed that if $f \in C^2$, then we can find $H$ as above which is additionally a $C^1$-diffeomorphism.

## 10. Lecture X: The Method of Lyapunov

10.1. **Lyapunov functions.** Having looked in some detail at hyperbolic critical points, and seeing that first order methods suffice to determine a wealth of information concerning the system in a neighbourhood of any such point, we turn now to a method that will shed some light on behviour near nonhyperbolic critical points.

First we shall refine our notions of stability. Let us call a fixed point $\mathbf{x}_0$ of the flow $\phi_t$ of an autonomous system STABLE if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $t \geq 0$,

$$\mathbf{y} \in B_\delta(\mathbf{x}_0) \quad \Longrightarrow \quad \phi_t(\mathbf{y}) \in B_\varepsilon(\mathbf{x}_0)\Big( = B_\varepsilon(\phi_t(\mathbf{x}_0))\Big),$$

otherwise the fixed point is UNSTABLE. This is a notion with which we are already familiar. Let us further say that $\mathbf{x}_0$ is ASYMPTOTICALLY STABLE if it is stable and if there exists a $\delta > 0$ such that in fact as $t \to \infty$, we have

$$\mathbf{y} \in B_\delta(\mathbf{x}_0) \quad \Longrightarrow \quad \lim_{t\to\infty} \phi_t(\mathbf{y}) = \mathbf{x}_0.$$

We know from any of the three preceding theorems on stable manifolds, centre manifolds, or the theorem of Hartman-Grobman that a hyperbolic critical point is either unstable, or otherwise asymptotically stable. However, fixed points for linear systems that exhibit centres, for example, are stable without being asymptotically stable.

The method of Lyapunov shall be able to help us to make a distinction between stable nonhyperbolic critical points that are asymptotically stable, and those which are not so.

**Theorem 10.1.** *Let $U$ be a neighbourhood of a fixed point $\mathbf{x}_0$ containing only one fixed point. Let $f \in C^1(U)$ determine an autonomous system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t)),$$

*and suppose that a function $V \in C^1(U)$ exists for which $V(\mathbf{y}) > V(\mathbf{x}_0)$ for $\mathbf{y} \in U$. If, furthermore,*

(i) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) \leq 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is stable;*
(ii) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) < 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is asymptotically stable; and*
(iii) $\mathrm{D}V|_{\mathbf{y}} \cdot f(\mathbf{y}) > 0$ *for every $\mathbf{y} \in U$, then $\mathbf{x}_0$ is unstable.*



Since the system is autonomous,

$$\frac{\mathrm{d}V \circ \mathbf{x}}{\mathrm{d}t} = \mathrm{D}V|_{\mathbf{x}} \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{D}V|_{\mathbf{x}} \cdot f(\mathbf{x}), \tag{21}$$

and so the conditions in the theorem statement are a condition on the increase or decreas of $V \circ \mathbf{x}(t)$ along a trajectory $\mathbf{x}(t) = \phi_t(\mathbf{y}_0)$. The applicability of the theorem depends on finding such a function $V$ and a neighbourhood $U$ for which $V$ and its derivatives has the requisite signs over the entire region $U$. A function $V$ satisfying (i) or (ii) is known as a LYAPUNOV FUNCTION of the system.

Given our discussion following the statement of the Centre Manifold Theorem (Thm.9.2) last time, we can see that if we can find a Lyapunov function around a non-hyperbolic fixed point of a nonlinear system, we shall be able differentiate between centres and foci.

Before we prove the theorem we require a result on continuous functions.

**Lemma 10.2.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $E \subseteq \mathbb{R}$ and $S \subseteq \mathbb{R}^d$ be a closed sets, furthermore let $S$ be bounded (i.e., contained within $B_R(\mathbf{0})$ for a large enough finite $R$). Then*

(i) *$V^{-1}(E)$ is closed in $\mathbb{R}^d$, and*
(ii) *$V$ attains its supremum in $S$ (i.e., $\exists \mathbf{w} \in S$ such that $V(\mathbf{w}) = \sup_{\mathbf{y}\in S} V(\mathbf{y})$).*

As in Lecture 8, by open sets we mean sets that can be made up of an arbitrary union of balls, and by closed sets we mean any set that can be written as a complement of an open set.

*Proof.* If $\{x_n\}$ is a sequence contained in $E$ that converges to $x$, it hold that $x \in E$ because otherwise $x$ is contained in some open ball $B_\delta$ outside of $E$, and this means $|x_n - x| > \delta$ for every $n$, no matter how large. This property therefore characterizes closed sets.

Now suppose $V^{-1}(E)$ is not closed. Then there is a sequence $\mathbf{y}_n \to \mathbf{y}$ for which $\mathbf{y}_n \in V^{-1}(E)$ for every $n$ but $\mathbf{y} \notin V^{-1}(E)$. Then $V(\mathbf{y}) \notin E$. Continuity transfers convergence, but $V(\mathbf{y})$ is not in $E$ so $V(\mathbf{y}_n)$ cannot converge to $E$. This is a contradiction on the continuity of $V$.

Let $m = \sup_{\mathbf{y}\in S} V(\mathbf{y})$. Consider the sequence $\{m - 1/n\}_{n=1}^{\infty}$. The inverse image of this sequence is a cloesd set $V^{-1}(\{m - 1/n\}_{n=1}^{\infty})$. Intersected with $S$, one obtains a bounded closed set. From each non-empty $V^{-1}(m - 1/n) \cap S$ choose a point $a_n$. Since $S$ is bounded, it can be halved by a codimension one subsapce so that infinitely many $\{a_n\}$ are in one half. This half can be halved again, ad infinitum, so that the sequence $\{a_n\}$ is seen necessarily to have a convergent subsequence. Since $S$ is close, this convergent subsequence converges to a point $b \in S$. For any $N > 0$, there exists an $M$ such that neighbourhood of $V(b)$ includes $m - 1/M$. This allows us to conclude by continuity that $V(b) = m$. □

*Proof of Thm. 10.1.* Without loss of generality, we can assume $V(\mathbf{x}_0) = 0$, and $V(\mathbf{y}) > 0$ for $\mathbf{y} \in U$ by adding a constant to $V$.

*Part (i):*

Since $U$ is open, there is a sufficiently small $\varepsilon$ such that $U$ contains the closed ball $\bar{B}_\varepsilon(\mathbf{x}_0)$. Since $V(\mathbf{y}) > 0$ on $U\backslash\{\mathbf{x}_0\}$, there is a positive minimum $m_\varepsilon > 0$ to the set

$$\{V(\mathbf{y}) : |\mathbf{y} - \mathbf{x}_0| = \varepsilon\} \subsetneq \mathbb{R}.$$

Since $V(\mathbf{x}_0) = 0$, by the continuity of $V$, for a sufficiently small $\delta < \varepsilon$, every $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ satisfies

$$V(\mathbf{y}_0) < m_\varepsilon.$$

By the non-increasing property of $V(\phi_t(\mathbf{y}_0))$, it holds that

$$V(\phi_t(\mathbf{y}_0)) \leq V(\mathbf{y}_0) < m_\varepsilon.$$

This ensures that $\phi_t(y_0)$ is never in the set $\{\mathbf{y} : |\mathbf{y} - \mathbf{x}_0| = \varepsilon\}$. From which we can conclude that $\phi_t(\mathbf{y}_0) \in B_\varepsilon(\mathbf{x}_0)$ for all time, and $\mathbf{x}_0$ is therefore a stable fixed point.

*Part (ii):*

From the above we know that if $\mathbf{y}_0 \in B_\delta(\mathbf{x}_0)$ for some sufficiently small $\delta$, $\phi_t(\mathbf{y}_0)$ remains in $B_\varepsilon(\mathbf{x}_0) \subseteq U$.

If $V(\phi_t(\mathbf{y}_0))$ is strictly decreasing along trajectories, and it is lower bounded by $V(\mathbf{x}_0) = 0$, it holds that $V(\phi_t(\mathbf{y}_0))$ must tend to a limit as $t \to \infty$. Suppose $V(\phi_t(\mathbf{y}_0)) \to m > 0$. Then for any $\eta > 0$, we can find a sufficiently large $T$ such that if $\tau > T$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} V(\phi_t(\mathbf{y}_0))\Big| < \eta.$$

We now derive a contradiction, the idea being that, supposing $V(\phi_t(\mathbf{y}_0))$ has to slow down to zero as it nears the set $\{\mathbf{y} : V($
$y) = m\}$, yet we can pick a point arbitrarily close to this set and show that the starting speed must be some magnitude uniformly bounded away from 0.

Observe that by continuity,

$$\mathcal{A} = \{\mathbf{y} : V(\mathbf{y}) = m\} \cap \bar{B}_\varepsilon(\mathbf{x}_0)$$

is closed and bounded, so that on this set, the continuous function (recall that $V \in C^1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} V(\phi_t(\mathbf{y}))$$

attains a minimum, say $\vartheta > 0$. Therefore, we can always find a $\delta'$ neighbourhood of this set on which

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} V \circ \phi_t > \vartheta'.$$

But we can choose $\eta < \vartheta'$ so that for large enough $\tau$, as $V(\phi_\tau(\mathbf{y}_0))$ approaches $\mathcal{A}$,

$$\left|\eta > \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=\tau} V(\phi_t(\mathbf{y}_0))\right| > \vartheta' > \eta,$$

a clear contradiction.

*Part (iii):*

Essentially reversing the signs/result of (ii).

□

10.2. **Examples.** As mentioned, applying the theorem turns on finding a Lyapunov function. There is no general way of doing so. We now look at two instructive examples.

**Example 10.1.** We consider the system

$$\begin{aligned}\dot{x} &= -2y + yz - x^3\\ \dot{y} &= x - xz - y^3\\ \dot{z} &= xy - z^3\end{aligned} \quad .$$

There is a fixed point at the origin. We can find $\mathrm{D}f(\mathbf{0})$:

$$\mathrm{D}f(\mathbf{0}) = \begin{pmatrix} 0 & -2 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_\pm = \pm 2i$. This is a non-hyperbolic fixed point for which the entire space is locally part of the centre manifold.

We consider the function $V(x, y, z) = x^2 + 2y^2 + z^2$. Setting $(x(t), y(t), z(t)) = \phi_t(\boldsymbol{\beta})$ for $\boldsymbol{\beta}$ in a sufficiently small neighbourhood of $\mathbf{0}$, the derivative can be computed as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(\phi_t(\boldsymbol{\beta})) &= \mathrm{D}V(x, y, z) \cdot f(x, y, z)\\ &= 2x(-2y + yz - x^3) + 4y(x - xz - y^3) + 2z(xy - z^3)\\ &= -2x^4 - 4y^4 - 2z^4\\ &< 0\end{aligned}$$

for $(x, y, z) \neq \mathbf{0}$.

Therefore the origin is asymptotically stable, even though it is not a sink, and we see that asymptotic stability does not necessarily imply the existence of a sink (stable focus or node) at non-hyperbolic critical points of nonlinear systems.

**Example 10.2.** Consider the second order equation

$$\ddot{x} + q(x) = 0.$$

We can reduce it to a first-order system:

$$\dot{x} = y, \qquad \dot{y} = -q(x).$$

This is known as a conservative system — essentially Newton's second law in which the force is a conservative force — it can be written as a gradient of a potential, or an "energy". For this reason it is also known as a gradient system, of which we shall see more in the next lecture. Such systems have Lyapunov functions that are easy to find if the energy as usually defined has a sign — the energy itself. We require $xq(x) > 0$, so that the force is restorative.

Suppose $q(x_0) = 0$, so that $(x_0, 0)$ is a fixed point. Set

$$V(x, y) = \frac{1}{2}y^2 + \int_{x_0}^{x} q(r)\, \mathrm{d}r.$$

Then $V(x, y) > 0$ in a neighbourhood of $(x_0, 0)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t), y(t)) = q(x)y + y(-q(x)) = 0.$$

Therefore the equilibrium $(x_0, 0)$ is stable.

## 11. Lecture XI: Gradient and Hamiltonian Systems

11.1. **Elementary Hamiltonian Dynamics.** A full discussion of Hamiltonian dynamics requires an introduction to manifolds with a symplectic structure, a sort of anti-symmetric inner product over the tangent bundle of spaces $T_pM$ as $p$ ranges over $M$. But it is nevertheless natural to give a superficial account of Hamiltonian systems after a discussion of the method of Lyapunov. We shall focus on select topics of the theory immediately pertainent to our ongoing discussion on fixed points.

A Hamiltonian system is a system over $\mathbb{R}^{2d}$, for which there is a function $H \in C^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ such that the system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\frac{\partial H}{\partial \mathbf{x}},$$

where $\mathbf{x}$ and $\mathbf{y}$ take values in $\mathbb{R}^d$. This function $H$ is known as the Hamiltonian (function).

The first thing to notice about Hamiltonian systems is that the Hamiltonian is a conserved quantity of the dynamics:

$$\frac{\mathrm{d}H(\mathbf{x},\mathbf{y})}{\mathrm{d}t} = \frac{\partial H}{\mathrm{d}\mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial H}{\mathrm{d}\mathbf{y}}\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = 0.$$

Therefore $H$ is a Lyapunov function for the corresponding Hamiltonian system if $H(\mathbf{y}) > H(\mathbf{x}_0)$ for every $\mathbf{y}$ in a neighbourhood of a fixed point $\mathbf{x}_0$, allowing us to conclude that that fixed point is stable. We have not excluded the possibility that we can find a better Lyapunov function that is strictly decreasing along trajectories. We shall be spending a little effort a little later on on telling when it is that a critical point is stable but not asymptotically stable.

Hamiltonian systems arise naturally in modelling physical systems in which no dissipative effects are present. The Hamiltonian of system is often interpretable as its total energy, or potential of some sort (gravitational potential, hydrolic potential, electric potential, etc.). The "classical mechanics way" to think about Hamiltonian systems is that $\mathbf{y} = \dot{\mathbf{x}}$, so that where $\mathbf{x}$ is the position of a particle, $\mathbf{y}$ is its (normalized mass) momentum. This is not always the case, of course, but where it holds true,

$$\ddot{\mathbf{x}} = -\frac{\partial H}{\mathrm{d}\mathbf{x}}.$$



We call systems of the form

$$\ddot{\mathbf{x}} = -f(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d$$

Newton systems if $f$ can be written as a gradient because they are Newton's second law with normalized mass and a conservative force given by $f(\mathbf{x})$. This is a very specific type of Hamiltonian system because if $f = \nabla_{\mathbf{x}} V$, then the Hamiltonian $H$ is given by

$$\frac{\partial H}{\mathrm{d}\mathbf{x}} = f(\mathbf{x}) = \nabla_{\mathbf{x}} V \implies H(\mathbf{x},\mathbf{y}) = V(\mathbf{x}) + G(\mathbf{y}),$$

and depends on the momentum variables $\mathbf{y}$ in a special way. The part dependent on $\mathbf{x}$ is the potential energy and the part dependent on $\mathbf{y}$ is the kinetic energy. In fact, taking

$$\nabla_{\mathbf{y}} G(\mathbf{y}) = \partial H/\partial \mathbf{y} = \dot{\mathbf{x}} = \mathbf{y},$$

we see that $G(\mathbf{y}) = |\mathbf{y}|^2/2$, as expected. With $d = 1$, we know we can always write $f$ as a gradient/derivative, simply by integrating $f$, but this is not always so for higher-dimensional $f$ because the integral of $f$ might not be path-independent.

11.2. **Planar Hamiltonian Systems.** The analysis of critical points for Hamiltonian systems for $d = 1$ is simpler also. We shall refine our analysis by giving geometric descriptions/definitions four types of behaviour around critical points in planar systems already familiar to us from the linear theory:

(i) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a FOCUS (or SPIRAL) if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $|\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)| \to \infty$ as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE).

(ii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a CENTRE if there exists a $\delta > 0$ such that every trajectory in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ is a closed curve.

(iii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autnomous system is a CENTRE-FOCUS if there exists an sequence of closed curves $\{\Gamma_n\}$ and a sequence of number $r_n \to 0$ such that $\Gamma_{n+1}$ is in the open set enclosed by $\Gamma_n$ and $\Gamma_n \subseteq B_{r_n}(\mathbf{x}_0)$, and every trajectory between $\Gamma_n$ and $\Gamma_{n+1}$ tends to one closed curve or the other as $t \to \pm\infty$. These closed curves $\Gamma_n$ are known as LIMIT CYCLES.

There must needs be an infinite sequence of limit cycles for this to be a genuinely new behaviour, because otherwise in a small enough neighbourhood of the fixed point, we see a focus, or a centre.

(iv) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a TOPOLOGICAL SADDLE if there exist two trajectories which approach $\mathbf{x}_0$ as $t \to \infty$, and two trajectories that approach $\mathbf{x}_0$ as $t \to -\infty$, and if there exists a $\delta > 0$ such that all other trajectories in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ leave $B_\delta(\mathbf{x}_0)$ as $t \to \pm\infty$. We call the four special trajectories SEPARATRICES.

We are ready to state a lemma:

**Lemma 11.1.** *If $(x_0, y_0)$ is a focus of the planar Hamiltonian system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\partial H}{\partial x},$$

*then $(x_0, y_0)$ is not a strict maximum or strict minimum of the Hamiltonian function $H$.*



As can be deduced from this theorem statement, Hamiltonian functions do not have to be Lyapunov functions in the strict sense we have required. However, the other case excluded by the statement is not at all exotic. If a function $V$ attains a maximum at a critical point $\mathbf{x}_0$, and $\mathrm{D}V \cdot f \geq 0$ in a neighbourhood of $\mathbf{x}_0$, then clearly $\mathbf{x}_0$ is stable because $-V$ would supply us with a suitable Lapunov function.

Since $H$ is conserved by the flow (invariant along trajectories), it is quite evident that if $H$ attained a strict maximum or strict minimum at the critical point $(x_0, y_0)$, and is $C^1$, then a trajectory cannot connect any points in a neighbourhood of $(x_0, y_0)$ with the critical point itself, for then $H$ would have to increase or decrease to the value it attains at the critical point. One instance in which $H$ might have a focus at a critical point, then, can be that $H$ vanishes to second order at the critical point.

In fact, this is the crux of the argument demonstrating the statement of the Lemma: If there is a focus, then there is a trajectory $\phi_t(u_0, v_0)$ that tends to the critical point $(x_0, y_0)$ such that $H(\phi_t(u_0, v_0))$ is constant. This means that $H$ cannot attain a strict maximm or minimum in any neighbourhood of $(x_0, y_0)$.

Before we state a more general theorem characterizing the behaviour of planar Hamiltonian systems around critical points, we shall impose two further requirements on the Hamiltonian functions of planar Hamiltonian systems. The first requirement that they be real analytic — that is, at any point, there is a neighbourhood such that $H$ can be expressed as a convergent series of its Taylor expansion about that point, and in particular, $H$ is smooth. There are smooth functions that are not real analytic — the most commonly cited example of which is

$$h(x) = \exp(1/x)\mathbb{1}_{[0,\infty)},$$

which is smooth not expressible as the convergent sum of its Taylor series in any neighbourhood around 0. We have seen that Lipschitz regularity gave us existence and uniqueness, and it should be no surprise that higher regularity/smoothness requirements yield similar dividends and rule out some behaviours that are difficult to analyse.

The second requirement is that $\nabla H$ is of full rank around $\mathbf{x}_0$. We say that the critical point $\mathbf{x}_0$ of a system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x}),$$

in $\mathbb{R}^d$ is NON-DEGENERATE if $\mathrm{D}f(\mathbf{x}_0)$ is non-degenerate/non-singular/does not have 0 as an eigenvalue. In the context of planar Hamiltonian systems, we see that this means the matrix

$$\mathrm{D}f = \begin{pmatrix} \partial^2 H/\partial y \partial x & \partial^2 H/\partial y^2 \\ -\partial^2 H/\partial x^2 & -\partial^2 H/\partial y \partial x \end{pmatrix}$$

is non-singular at a critical point $(x_0, y_0)$. Notice that this matrix can be obtained from the Hessian $\nabla^2 H$ of $H$ by left multiplication with

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which implies that $\det(\mathrm{D}f) = \det(\nabla^2 H)$. That is, we require that $H$ does not vanish to second order.

From elementary calculus, if $H \in C^2(\mathbb{R}^2)$, then at a point $(x_0, y_0)$ for which $\nabla H = (0,0)^\top$, one can deduce if $(x_0, y_0)$ is a maximum, a minimum, or a saddle point according as $\mathrm{D}f(x_0, y_0)$ is negative definite, positive definite, or has both positive and negative eigenvalues. In particular, if $H$ has a saddle point at $(x_0, y_0)$, then $\det(\nabla^2 H)(x_0, y_0) < 0$.

We state now our first result on concrete behaviour of trajectories near nonhyperbolic critical points for general planar autonomous systems:

**Theorem 11.2.** *Let $U \subseteq \mathbb{R}^2$ be a neighbourhood of a critical point $\mathbf{x}_0$ of an autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(U)$. Suppose $\mathbf{x}_0$ is a centre for the linearized dynamics. Then $\mathbf{x}_0$ is either a centre, a centre-focus, or a focus for the original autonomous system.*



We shall defer this proof to the next lecture. Meanwhile we state another result due in part to Dulac which we shall discuss a few more lectures thence:

**Theorem 11.3** (Dulac)**.** *In any bounded region of the plane, an analytic planar system has at most a finite number of limit cycles.*

This readily implies that for analytic $f$, the autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$ cannot have centre-foci. Lemma 11.1 also rules out foci under certain situations. This brings us to the main theorem of this lecture:

**Theorem 11.4.** *Let $\mathbf{x}_0$ be a nondegenerate critical point of a planar analytic Hamiltonian system. Then $\mathbf{x}_0$ is a topological saddle of the system if it is a saddle for the Hamiltonian function, and $\mathbf{x}_0$ is a centre if it is a strict local maximum or minimum for the Hamiltonian function.*

The non-degeneracy and strict optimum conditions rule out focus behaviour by Lemma 11.1, and the analyticity assumption implies that a centre for the linearized system is a centre for the full system via Thm. 11.2 and Dulac's Theorem. Saddles are hyperbolic, and the the statement in the theorem pertaining to them follows from the Hartman-Grobman Theorem.

11.3. **Gradient Systems.** Since Hamiltonian systems arise naturally in modelling potentials, it is of interest also to consider the orthogonal system. Given the Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = -\frac{\partial H}{\partial \mathbf{x}},$$

the ORTHOGONAL SYSTEM is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\partial H}{\partial \mathbf{x}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \frac{\partial H}{\partial \mathbf{y}},$$

which we can also write as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = \nabla H. \tag{22}$$

Since

$$\nabla H \cdot \Big(\frac{\partial H}{\partial \mathbf{y}}, -\frac{\partial H}{\partial \mathbf{x}}\Big)^{\top} = 0,$$

the trajectories of the orthogonal system are orthogonal to the trajectories of the Hamiltonian system. Where the trajectories of the Hamiltonian system are iso-energetic lines, the orthogonal trajectories run along the direction of the steepest change of $H$, that is, along $\nabla H$. If $H$ were of a form giving a Newton system, the $\mathbf{x}$ trajectories would be the field lines of the "force" under the energy potential given by $H$. Systems of the form (22) are known as GRADIENT SYSTEMS.

Since trajectories of the gradient system are always orthogonal to the level sets of $H$, we can use $H$ (or $-H$) as a Lyapunov function and formulate and show the "dual" theorem to Thm.11.4:

**Theorem 11.5.** *Let $\mathbf{x}_0$ be a nondegenerate critical point of a planar analytic gradient system with potential $H$. Then $\mathbf{x}_0$ is a topolgocial saddle for the gradient system if it is a saddle for $H$ and it is a stable or unstable node of the system according as it is a strict local maximum or strict local minimum for $H$.*



**Example 11.1** (Simple pendulum)**.** The simple one-dimensional (i.e., one degree of freedom) pendulum with a massless string under the effect of gravity only can be modelled as

$$\ddot{\vartheta} + \sin(\vartheta) = 0.$$

This system is usually linearized by considering the small angle approximation $\sin(\vartheta) \approx \vartheta$ when $\vartheta$ is small.

The full system can be readily analysed however, as it is a Newton system:

$$\dot{\vartheta} = \psi, \qquad \dot{\psi} = \sin(\vartheta).$$

There are fixed points at $\psi = 0$ and $\vartheta \in \pi\mathbb{Z}$.

The potential energy/Hamiltonian in this case is

$$H(\vartheta) = \int_0^{\vartheta} \sin(\eta)\,\mathrm{d}\eta = 1 - \cos(\vartheta).$$

As can be computed with the Hessian of $H$, via Thm. 11.4, the fixed points at $2\pi\mathbb{Z}$ are centres and the fixed points at $(2n+1)\pi\mathbb{Z}$ are saddles.

## 12. Lecture XII: Critical Points of Planar Systems I

Let us continue our discussion on critical points of planar systems in greater generality than we have done in the previous lecture. In this lecture we are still interested in the four types of nondegenerate behaviours that are manifested in planar linear systems — namely, centres, foci, nodes, and saddles, and perturbations of them by nonlinearities. Recall that we have also defined the centre-focus which does not occur in linear or even analytic systems. We shall see in the next lecture that nonlinear systems can manifest behaviours vastly different from these five.

With reference to the theorems that shall follow, by an ISOLATED CRITICAL POINT, we mean a critical point $\mathbf{x}_0$ for which there exists a $\delta > 0$ such that $B_\delta(\mathbf{x}_0)$ does not contain any other critical points. Before we begin let us also review and expand the list of fixed points of nonlinear systems we considered last time:

(i) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a NODE if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)$ tends to a finite limit as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE). It is a PROPER NODE if each ray from $\mathbf{x}_0$ is tangent to some trajectory.

(ii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a FOCUS (or SPIRAL) if there exists a $\delta > 0$ such that for $\mathbf{y}_0$ with $0 < |\mathbf{x}_0 - \mathbf{y}_0| < \delta$, $|\phi_t(\mathbf{y}_0) - \mathbf{x}_0| \to 0$ and $|\arg(\phi_t(\mathbf{y}_0) - \mathbf{x}_0)| \to \infty$ as $t \to \infty$ (STABLE) or as $t \to -\infty$ (UNSTABLE).

(iii) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a CENTRE if there exists a $\delta > 0$ such that every trajectory in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ is a closed curve.

(iv) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autnomous system is a CENTRE-FOCUS if there exists an sequence of closed curves $\{\Gamma_n\}$ and a sequence of number $\delta_n \to 0$ such that $\Gamma_{n+1}$ is in the open set enclosed by $\Gamma_n$ and $\Gamma_n \subseteq B_{\delta_n}(\mathbf{x}_0)$, and every trajectory between $\Gamma_n$ and $\Gamma_{n+1}$ tends to one closed curve or the other as $t \to \pm\infty$. These closed curves $\Gamma_n$ are known as LIMIT CYCLES.

(v) A critical point $\mathbf{x}_0 \in \mathbb{R}^2$ for an autonomous system is a TOPOLOGICAL SADDLE if there exist two trajectories which approach $\mathbf{x}_0$ as $t \to \infty$, and two trajectories that approach $\mathbf{x}_0$ as $t \to -\infty$, and if there exists a $\delta > 0$ such that all other trajectories in $B_\delta(\mathbf{x}_0)\backslash\{\mathbf{x}_0\}$ leave $B_\delta(\mathbf{x}_0)$ as $t \to \pm\infty$. We call the four special trajectories SEPARATRICES.



First, we present a general theorem due to Bendixon, whose work shall be the topic of further lectures in this module:

**Theorem 12.1** (Bendixson)**.** *Let $(x_0, y_0) \in \mathbb{R}^2$ be an isolated critical point of a $C^1$-first order autonomous system, then either (i) every neighbourhood of the critical point contains a closed trajectory, or (ii) there exists a trajectory that tends to $(x_0, y_0)$ as $t \to \pm\infty$.*

This may seem a rather vacuous theorem, and indeed it can be verified quite readily if one attempts to draw trajectories near an isolated fixed point. We shall not spend time proving it.

Recall that from the Hartman-Grobman theorem, around a hyperbolic critical point $\mathbf{x}_0 \in \mathbb{R}^d$ of a $C^1$-autonomous system, we can always find neighbourhoods $U$ and $V$ and a homeomorphism $\psi : U \to V$ such that the flow was homeomorphic to the linear flow:

$$\psi \circ \phi_t = \exp(\mathrm{D}f(\mathbf{x}_0))\psi.$$

It turns out that this simply continuous homeomorphism is capable of turning a node into a focus. But if $\psi$ is a $C^1$-diffeomorphism, this is not possible. Happily, as we also mentioned in remarks following the Hartman-Grobman Theorem, Hartman also showed that if the system is a $C^2$-first order autonomous system, a homeomorphism that is additionally a $C^1$-diffeomorphism exists. To see what is happening, consider the topologist's sine curve, $y = \sin(1/x)$, which is continuous but has unbounded derivative closer and closer to the origin.

From this we can conclude that

**Theorem 12.2.** *Let $(x_0, y_0) \in \mathbb{R}^2$ be a hyperbolic fixed point of a $C^2$-first order autonomous system. Then $(x_0, y_0)$ is a (topological) saddle, a stable or unstable focus, or a stable or unstable node, according as $(x_0, y_0)$ is a saddle, a stable or unstable focus, or a stable or unstable node of the linearized system, and conversely.*

The correspondence between the full system and the linearized system in the saddle case can be had for $C^1$-first order autonomous systems as well. Let us look at one example where the system is only $C^1$ and the conclusion of the theorem above does not hold.

**Example 12.1.** Consider the system with $f$ defined by the following for $(x, y) \neq (0, 0)$:

$$\dot{x} = -x - \frac{y}{\log(\sqrt{x^2+y^2})},$$
$$\dot{y} = -y + \frac{x}{\log(\sqrt{x^2+y^2})}.$$

We complete this definition by setting $f((0,0)) = (0,0)$, so that the origin is a fixed point by fiat. In polar coordinates, the system becomes

$$\dot{r} = -r, \qquad \dot{\vartheta} = 1/\log(r).$$

Recall that a change of coordinates is really a local homeomorphism.

This nonlinear system can be readily integrated:

$$r(t) = r_0 e^{-t}, \qquad \vartheta(t) = -\log\left(1 - t/\log(r_0)\right) + \vartheta_0.$$

This shows that starting in a small enough neighbourhood of the origin, i.e., $r_0 < 1$ we have $|r| \to 0$ and $|\vartheta| \to \infty$ as $t \to \infty$, and the origin is a stable node.

The linearized system is determined by

$$\mathrm{D}f = \begin{pmatrix} -1 & -(\log(\sqrt{x^2+y^2}))^{-1} \\ (\log(\sqrt{x^2+y^2}))^{-1} & -1 \end{pmatrix} + (\log(\sqrt{x^2+y^2}))^{-2}(x^2+y^2)^{-1} \begin{pmatrix} xy & y^2 \\ -x^2 & -xy \end{pmatrix},$$

and

$$\mathrm{D}f((0,0)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which yields a stable node.

Changing to polar coordinates in planar systems may be the main technique introduced in this lecture.

The aim of this lecture is to show the following theorem which underpinned our deductions last time:

**Theorem 12.3.** *Let $U \subseteq \mathbb{R}^2$ be a neighbourhood of a critical point $\mathbf{x}_0$ of an autonomous system $\dot{\mathbf{x}} = f(\mathbf{x})$, $f \in C^1(U)$. Suppose $\mathbf{x}_0$ is a centre for the linearized dynamics. Then $\mathbf{x}_0$ is either a centre, a centre-focus, or a focus for the original autonomous system.*

*Proof.* By a translation if necessary, let the critical point $\mathbf{x}_0$ be the origin. If the origin is a centre, we know that by a change of variables, we can write $\mathrm{D}f$ as

$$\mathrm{D}f = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix},$$

which has eigenvalues $\lambda_\pm = \pm \imath b$.

The full planar system is then

$$\dot{x} = -bx + p(x, y)$$
$$\dot{y} = by + q(x, y).$$

Since $(0,0)$ is a fixed point, $p(0,0) = q(0,0) = 0$.

Recall our decomposition of a general $C^1$ system in our discussion of the Stable Manifold Theorem. For every $\delta > 0$, there is an $\varepsilon > 0$ such that if $|(x, y)^\top| < \delta$,

$$|(p(x, y), q(x, y))^\top| < \varepsilon |(x, y)^\top|.$$

This means that $p = o(r)$ and $q = o(r)$.

Now

$$2r\dot{r} = \frac{\mathrm{d}}{\mathrm{d}t} r^2 = 2x\dot{x} + 2y\dot{y} = -2xyb + 2xyb + 2xp + 2yq = o(r^2),$$
$$\dot{\vartheta} = \frac{\mathrm{d}}{\mathrm{d}t} \arctan(y/x) = \frac{1}{1 + (y/x)^2} \left( \frac{1}{x}\dot{y} - \frac{y}{x^2}\dot{x} \right) = b + o(1),$$

as $r \to 0$, and we have

$$\dot{r} = o(r), \qquad \dot{\vartheta} = b + o(1).$$

This means that for sufficiently small $r_0$, say $r_0 < \delta$, we find

$$\dot{\vartheta} \geq b/2, \qquad \vartheta \geq bt/2 + \theta_0,$$

as $t \to \infty$, and $\vartheta$ is a monotonically increasing function of $t$. This means $\vartheta$ is invertible for small enough $r$, and so it makes sense to write the radius $\tilde{r} = r \circ \vartheta^{-1}$ as a function of the angle/argument.

Suppose the origin is neither a centre nor a centre focus. Then for $\delta > 0$ small enough, there are no closed trajectories in $B_\delta(\mathbf{0}) \backslash \{\mathbf{0}\}$. Without loss of generality (otherwise use time reversal $t \mapsto -t$) we can take

$$\tilde{r}(\vartheta_0 + 2\pi) < \tilde{r}(\vartheta_0).$$

This argument can be iterated and in order to avoid trajectories that cross (which cannot happen in autonomous systems), we have

$$\tilde{r}(\vartheta_0 + 2k\pi) < \tilde{r}(\vartheta_0 + 2(k-1)\pi)$$

for every $k \in \mathbb{N}$.

Therefore the sequence of numbers in $k$ is monotonically decreasing and lower bounded by 0, which means there is a number $\varrho$ such that

$$\varrho = \lim_{k \to \infty} \tilde{r}(\vartheta_0 + 2k\pi).$$



Now we are going to consider the convergence of the functions $r_k(\vartheta) = \tilde{r}(\vartheta_0 + 2k\pi + \vartheta)$ on $[0, 2\pi]$. We shall show that it is uniformly bounded and equicontinuous, whereby the Arzela-Ascoli Theorem will allow us to conclude that $r_k(\vartheta)$ converges to a continuous function $R(\vartheta)$ on $[0, 2\pi]$ in the continuous/uniform norm that we have seen before. Since $R(\vartheta)$ will then be arbitrarily cloes to $R(\vartheta + 2\pi)$ for any $\vartheta \in [0, 2\pi]$, $R$ will be non-zero periodic function, and a closed trajectory, a contradiction.

We already know that the functions $r_k$ are uniformly bounded :

$$\varrho \leq r_k(\vartheta) \leq \sup_{\theta \in [0, 2\pi]} \tilde{r}(\vartheta_0 + \theta).$$

We also know that

$$\begin{aligned} \frac{\mathrm{d}\tilde{r}}{\mathrm{d}\vartheta} = \frac{\dot{\tilde{r}}}{\dot{\vartheta}} &= \frac{(x\dot{x} + y\dot{y})/\tilde{r}}{(x\dot{y} - y\dot{x})/\tilde{r}^2} \\ &= \frac{p(x, y)\cos(\vartheta) + q(x, y)\sin(\vartheta)}{(\cos(\vartheta)q(x, y) - \sin(\vartheta)p(x, y))/\tilde{r}} \\ &\leq \frac{M}{b/2}. \end{aligned}$$

This shows that $r_k$ are equicontinuous. And thus is our theorem proven.

□

## 13. Lecture XIII: Critical Points of Planar Systems II

We shall now be looking into some dynamics around critical points that are not even qualitatively similar to any dynamics that can be observed around critical points of linear systems. From the Hartman-Grobman theorem, we know that this means we cannot be working at hyperbolic critical points. We shall be looking at some further dynamics on the centre manifold that are neither centres nor foci.

To circumscribe our discussion, we shall consider only autonomous *analytic* systems around isolated critical points. From Thm. 12.3 and Dulac's Theorem (Thm. 11.3), we know that we are not considering critical points for which the linearized system exhibits a centre. We are also not interested in critical points of systems for which the linearization exhibits nodes, foci, or saddles, as analytic systems are immediately $C^2$. Since we are working on real planar systems, this leaves us with critical points with zero eigenvalues in the linearization.

13.1. **Degenerate linear planar systems.** We know that critical points of degenerate linear systems are never isolated. Herein is the crux of the matter. In considering higher powers beyond the first order approximation, we can "unfold" a degenerate linearized system into exhibiting a wide variety of different behaviours.

First we take another look at degenerate linear planar systems, which we have only done in obiter.

There are three ways that a linear planar system can be degenerate. By the Jordan Normal Form Theorem (Thm. 3.1), we can assume that the linear system has been put into Jordan normal form. We can have systems governed by matrices with one zero eigenvalue, two zero eigenvalues but with geometric multiplicity one, or two zero eigenvalues with a geometric multiplicity of two, respectively:

$$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

for $\lambda \in \mathbb{R}\backslash\{0\}$.

As we have mentioned, the final case is immediate — the entire $\mathbb{R}^2$ are fixed points. The general solution to systems governed by the first matrix is

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution to systems governed by the second matrix is not too much more difficult to write down, recallling the Jordan chain procedure:

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = (C_1 + C_2 t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

As we have already mentioned in Lecture 9, these rather tame behaviours are much more readily perturbed by higher order terms than when the governing matrix is not degenerate.

13.2. **Nonhyperbolic critical points of planar systems.** Next we are obliged to first consider a theorem about analytic first order systems:

**Theorem 13.1.** *Consider the autonomous analytic planar system given by*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$

*Let $P_m(x, y)$ and $Q_m(x, y)$ be the $m$-th degree polynomials in the Taylor expansions of $P$ and $Q$ about an isolated fixed point at the origin. Any trajectory that approaches the origin as $t \to \infty$ either spirals towards $\mathbf{0}$ (i.e., argument tends to infinity) or approaches it along a definite direction $\theta_0$ as $t \to \infty$.*

*If one trajectory spirals towards $\mathbf{0}$, then in a deleted neighbourhood $B_\delta(\mathbf{0})\backslash\{\mathbf{0}\}$, all trajectories spiral towards $\mathbf{0}$.*

*If there is a definite direction of approach, then either*

$$\forall m \geq 1, \quad xQ_m(x,y) - yP_m(x,y) \equiv 0,$$

*or along any direction $\theta_0$ of approach,*

$$\forall m \geq 1, \quad \cos(\theta_0)Q_m(\cos(\theta_0),\sin(\theta_0)) - \sin(\theta_0)P_m(\cos(\theta_0),\sin(\theta_0)) = 0.$$

We have already looked at centres and foci, and mentioned that centre-foci cannot appear for analytic planar systems.

The expressions in the theorem statements come from the angular velocity in polar coordinates:

$$\dot{\vartheta} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2} = \frac{xP - yQ}{r^2};$$

they are also two-dimensional versions of the cross product,

$$(x,y)^\top \wedge (P_m(x,y), Q_m(x,y))^\top.$$

It is a result of calculus that vector fields can be decomposed into divergence-free and curl-free parts. Another way of saying this is that for any fixed vector in $\mathbb{R}^d$, the subspace of vectors orthogonal to it and the subspace of vectors that has zero wedge/cross product with it are othodgonal and decompose $\mathbb{R}^d$. The result above says that unless there is a spiral, any vector field along which an approaching trajectoy approaches must be radial to arbitrary degree — that is, purely "divergence", and and any non-radial component of the approach come from the constant terms of the Taylor expansion.

It is then clear that the first non-spiral approaching possibility allowed by the theorem statement are nodes, whilst two of the the separatrices of a saddle fall under the second non-spiral approaching possibility allowed by the theorem statement as $t \to \infty$, and the remaining two are included in the same provision as $t \to -\infty$.

But the theorem also allows for more diverse behaviours under its second non-spiralling provision. It is in fact possible, as shall be demonstrated in computable examples later, that the plane gets divided into sectors separated much like saddles by separatrices that approach the critical point along definite directions as $t \to \infty$ or as $t \to -\infty$. We denominate three possible types of sectors as being HYPERBOLIC, PARABOLIC, ELLIPTIC according as there is a small enough neighbourhood about the critical point such that each trajectory in the sector not including the separatrices

(i) leaves the neighbourhood as $t \to \pm\infty$, or
(ii) leaves the neighbourhood as $t \to \infty$ and approaches the critical point as $t \to -\infty$, or vice versa, or
(iii) approaches the critical point as $t \to \pm\infty$.

The saddle is then seen to be a critical point with four separatrices and four hyperbolic sectors, and a node is a critical point with one single parabolic sector.

As mentioned at the beginning of this lecture, from the Hartman-Grobman Theorem, or more directly, from Thm.12.2, we know that other sectoring behaviours are not exhibited at hyperbolic fixed points. We also have considered behaviours around critical points for which the linearized system exhibits centres, and Thm. 12.3 ensures that in the analytic case, the full dynamics about these critical points are centres or foci. This leaves critical points with one or two eigenvalues that are zero.

*13.2.1. One zero eigenvalue.*

We first consider systems with one eigenvalue set to nought. Again, from the theorem on Jordan normal forms (Thm. 3.1), we know that the linearized system is governed by

$$\mathrm{D}f = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

By scaling, we may take $\lambda = 1$, without loss of generality. If we desire that $\lambda$ correspond to a stable subspace, we simply consider the system backwards in time. This compels us to consider systems of the type:

$$\begin{aligned}\dot{x} &= P(x, y)\\ \dot{y} &= y + Q(x, y),\end{aligned}$$

where $P$ and $Q$ vanish to second order around the *isolated* fixed point $(x_0, y_0)^\top = \mathbf{0}$.

From the implicit function theorem (Thm.8.4), there is a function $\phi$ such that $y = \phi(x)$ solves $y + Q(x, y) = 0$ in a neighbourhood of $\mathbf{0}$. This is, locally, graph of the nullcline. We can write $\phi(x)$ as

$$\phi(x) = \phi(0) + \phi'(0)x + \cdots,$$

as we are in an analytic setting. Furthermore, since $P$ is analytic and vanishes to second order in a neighbourhood of $\mathbf{0}$, we can write

$$\psi(x) = P(x, \phi(x)) = \sum_{m\geq 2} a_m x^m.$$

It turns out that the lowest order term of $\psi$ — $P$ evaluated on the other nullcline — gives us further information beyond the linearization that can already classify the behaviour of the system in a neighbourhood of the critical point.

First we need to look at some classes of behaviours:

(i) a critical point is a CRITICAL POINT WITH AN ELLIPTIC DOMAIN if it has four separatrices, one elliptic sector, one hyperbolic sector, and two parabolic sectors;
(ii) a critical point is a SADDLE-NODE if it has three separatrices, two hyperbolic sectors, and one parabolic sector; and
(iii) a critical point is a CUSP if it has two separatrices and two hyperbolic sectors.

And we have the following theorem:



**Theorem 13.2.** *Let $\psi(x) = P(x, \phi(x)) = \sum_{m\geq 2} a_m x^m$ be defined as before in a neighbourhood of the origin for the planar system governed by $P(x, y)$ and $Q(x, y)$, also previously defined. Let $\ell$ be the smallest integer for which $a_\ell \neq 0$.*

(i) *If $\ell \equiv 1 \,(\mathrm{mod}.2)$ and $a_\ell > 0$, then $\mathbf{0}$ is an unstable node,*
(ii) *if $\ell \equiv 1 \,(\mathrm{mod}.2)$ and $a_\ell < 0$, then $\mathbf{0}$ is a (topological) saddle, and*
(iii) *if $\ell \equiv 0 \,(\mathrm{mod}.2)$, then $\mathbf{0}$ is an saddle-node.*

*13.2.2. Two zero eigenvalues with unit geometric multiplicity.*

The corresponding theorem becomes considerably more complicated when both eigenvalues are zero, but the dimension of $\ker(\mathrm{D}f)$ is only one. We mention this for completeness.

In this case, it turns out that there is always a change-of co-ordinates that leaves the system in the following normal form:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= a_k x^k + b_n x^n y + O(x^{k+1} + x^{n+1} y + y^2)\end{aligned}$$

where the higher order terms are dominated by the first two terms in a neighbourhood of the fixed point $\mathbf{0}$.

We define two further parameters:

$$m := [k/2], \qquad \lambda := b_n^2 + (2k + 2)a_k.$$

The behaviours exhibited at the fixed point obey the following schematic:

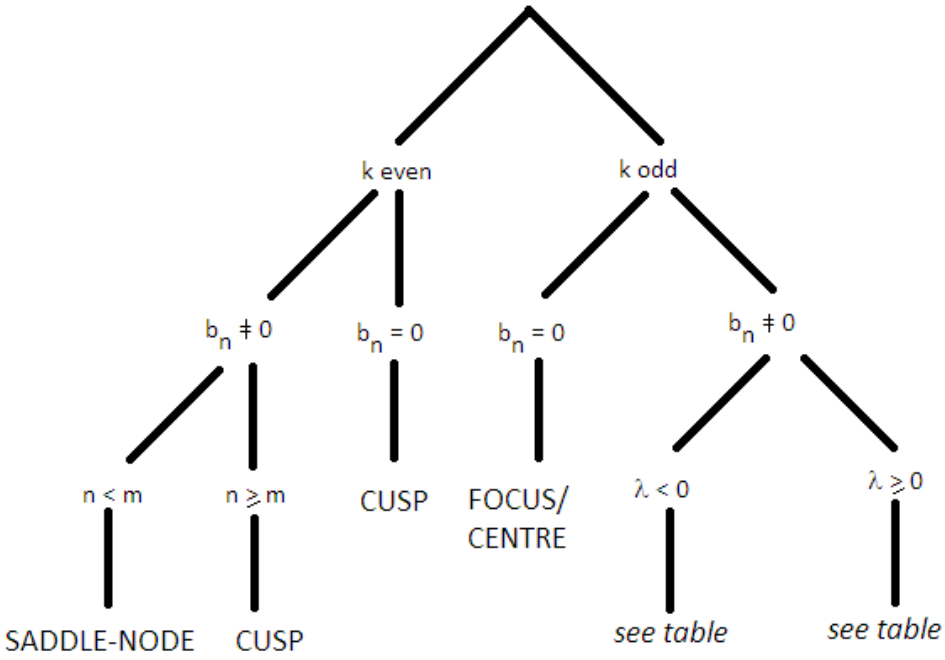



FIGURE 2. behaviours at critical points

The schematic above must also be supplemented by the following tables:
For $\lambda < 0$,

| $b_n \neq 0,\ \lambda < 0$ | $n < m$ | $n = m$ | $n > m$ |
|---|---|---|---|
| $n$ even | node | focus/centre | focus/centre |
| $n$ odd | elliptic domain | focus/centre | focus/centre |

and for $\lambda \geq 0$,

| $b_n \neq 0,\ \lambda \geq 0$ | $n < m$ | $n = m$ | $n > m$ |
|---|---|---|---|
| $n$ even | node | node | focus/centre |
| $n$ odd | elliptic domain | elliptic domain | focus/centre |

.

[There is no expectation that this schematic and its associated tables be committed to memory.]

*13.2.3. Two zero eigenvalues with geometric multiplicity of two.*

In this case, there are very few general theorems, and the behaviour on the centre manifold can be very complicated. In particular, if $P$ and $Q$ both vanish to order $m$ around $\mathbf{0}$, then the plane is locally split into $(2m+1)$ sectors. We shall look at this case in greater detail in our discussion on index theory.

13.3. **Examples.**

**Example 13.1.** The system

$$\dot{x} = x^2, \qquad \dot{y} = y,$$

has a linearization governed by

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and so has only one zero eigenvalue. It falls into case (iii) of Thm. 13.2. Therefore we expect a saddle-node.

**Example 13.2.** The system

$$\dot{x} = y, \qquad \dot{y} = -x^3 + 4xy,$$

has a linearization governed by

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and so has two zero eigenvalue, and $\dim\ker(\mathrm{D}f) = 1$. We find that $k = 3$, $m = 1$, $n = 1$, $a_3 = -1$, $b_1 = 4 \neq 0$, and $\lambda = b_1^2 - (2k+2)a_k = 24 > 0$. Consulting the schematic, we expect a critical point with an elliptic domain.

## 14. Lecture XIV: Centre Manifold Theory

14.1. **The Local Centre Manifold Theorem.** In this halfway lecture, we shall discuss a landmark structure theorem in our understanding of the local theory of first order autonomous systems, a culmination of our work so far. We shall turn our attention back to the problem in $\mathbb{R}^d$. We begin as before by noting that our first-order approach was based on the observation that around an isolated fixed point at the origin, we can decompose a general $C^1$-first-order autonomous system into

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{C}\mathbf{x} + F(\mathbf{x},\mathbf{y},\mathbf{z})\\ \dot{\mathbf{y}} &= \mathbf{P}\mathbf{y} + G(\mathbf{x},\mathbf{y},\mathbf{z}),\\ \dot{\mathbf{z}} &= \mathbf{Q}\mathbf{z} + H(\mathbf{x},\mathbf{y},\mathbf{z})\end{aligned} \tag{23}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ taking values in $\mathbb{R}^c$, $\mathbb{R}^s$, and $\mathbb{R}^u$, respectively, where $c+s+u=d$, and $\mathbf{C}\in\mathbb{C}^{c\times c}$ has $c$ eigenvalues with zero real parts, $\mathbf{P}\in\mathbb{C}^{s\times s}$ has $s$ eigenvalues with negative real parts, and $\mathbf{Q}\in\mathbb{C}^{u\times u}$ has $u$ eigenvalues with positive real parts. All eigenvalues are counted with multiplicities, and complex eigenvalues come in conjugate pairs. Finally, $F(\mathbf{0})=G(\mathbf{0})=H(\mathbf{0})=\mathbf{0}$.

Recall (Lecture 9) that two systems on $\mathbb{R}^d$ with flows $\phi_t$ and $\psi_t$ and fixed points $\mathbf{x}_0$ and $\mathbf{y}_0$ are TOPOLOGICALLY CONJUGATE if there exist neighbourhoods $U$ of $\mathbf{0}$ and $V$ of $\mathbf{0}$, and a homeomorphism $h: U\to V$ for which $h(\mathbf{0})=\mathbf{0}$ and

$$\psi_t\circ h = h\circ\phi_t.$$

The stable manifold theorem (Thm.9.1) and the Hartman-Grobman theorem (Thm.9.3) ensure us that in a neighbourhood of a hyperbolic fixed point, a $C^1$-first order autonomous system is topologically conjugate to its linearization. The centre manifold theorem (Thm.9.2) established the existence of a local centre manifold in addition to the stable and unstable manifolds described by the stable manifold theorem. All three local manifolds are invariant manifolds under the flow of the dynamical system. For planar systems, we have also established some very specific results regarding nonhyperbolic fixed points. We shall now put these together in the the following:



**Theorem 14.1** (Local Centre Manifold Theorem)**.** *On a neighbourhood $U\subseteq\mathbb{R}^d$ about the origin, there exist functions $h_1: B_\delta(\mathbf{0})\subseteq\mathbb{R}^c\to B_{\delta'}(\mathbf{0})\subseteq\mathbb{R}^s$ and $h_2: B_\delta(\mathbf{0})\subseteq\mathbb{R}^c\to B_{\delta'}(\mathbf{0})\subseteq\mathbb{R}^u$, satisfying*

$$\begin{aligned}\mathrm{D}h_1(\mathbf{w})(\mathbf{C}\mathbf{w}+F(\mathbf{w},h_1(\mathbf{w}),h_2(\mathbf{w})))-\mathbf{P}h_1(\mathbf{w})-G(\mathbf{w},h_1(\mathbf{w}),h_2(\mathbf{w}))=0,\\ \mathrm{D}h_2(\mathbf{w})(\mathbf{C}\mathbf{w}+F(\mathbf{w},h_1(\mathbf{w}),h_2(\mathbf{w})))-\mathbf{Q}h_2(\mathbf{w})-H(\mathbf{w},h_1(\mathbf{w}),h_2(\mathbf{w}))=0,\end{aligned} \tag{24}$$

*for all $\mathbf{w}\in B_\delta(\mathbf{0})\subseteq\mathbb{R}^c$, such that the system (23) is topologically conjugate to*

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{C}\mathbf{x}+F(\mathbf{x},h_1(\mathbf{x}),h_2(\mathbf{x}))\\ \dot{\mathbf{y}} &= \mathbf{P}\mathbf{y}\\ \dot{\mathbf{z}} &= \mathbf{Q}\mathbf{z}\end{aligned},$$

*in a neighbourhood of $\mathbf{0}$. If $F,G,H\in C^r(U)$, then $h_1$ and $h_2$ can be chosen to be $r$-times continuously differentiable.*

This result, sometimes instead of Thm.9.2, and sometimes in conjunction with it, is often known simply as the "Centre Manifold Theorem".

The functions $h_1$ and $h_2$ are locally the equations that determine the centre manifold via:

$$W^c(\mathbf{0})=\{(\mathbf{x},\mathbf{y},\mathbf{z})\in\mathbb{R}^d:\quad \mathbf{y}=h_1(\mathbf{x}),\ \mathbf{z}=h_2(\mathbf{x})\}.$$

The equations (24) then come from the dynamics on the centre manifold given by

$$\begin{aligned}\dot{\mathbf{y}} &= \mathrm{D}h_1(\mathbf{x})\dot{\mathbf{x}}\\ \dot{\mathbf{z}} &= \mathrm{D}h_2(\mathbf{x})\dot{\mathbf{x}},\end{aligned}$$

which gives us the evolution of $\mathbf{y}$ and $\mathbf{z}$ restricted to the centre manifold, and possibly reminescent of (21). The theorem above says that on the hyperbolic "parts" of the system, we can approximate

the full dynamics relatively well by their linearization, but on the centre "part" we cannot simply consider the subspace/first-order approximation. It neverthelss allows us to describe dynamics in $\mathbb{R}^d$ very precisely, given what we already know about planar dynamics, if the centre manifold is of dimension two.

The equations (24) can be impossible to solve exactly for $h_1$ and $h_2$, being a nonlinear coupled system of partial differential equations. Nevertheless if $r$ is sufficiently large, we can approximate $h_1$ and $h_2$ by power series to a high degree of accuracy, allowing us to make higher-order approximations to the dynamical system.

**Example 14.1.** Let us consider another dynamical system in a neighbourhood of an isolated fixed point for which the centre manifold is locally planar:

$$\begin{aligned}\dot{x}_1 &= x_1 y - x_1 x_2^2 = F_1(x_1, x_2, y)\\ \dot{x}_2 &= x_2 y - x_1^2 x_2 = F_2(x_1, x_2, y)\\ \dot{y} &= -y + x_1^2 + x_2^2 = -y + G(x_1, x_2, y).\end{aligned}$$

Here, $c = 2$ and $s = 1$, $\mathbf{C}$ is the $2 \times 2$ zero matrix, and $\mathbf{P}$ is the $1 \times 1$ matrix $-1$. The centre subspace at the fixed point $\mathbf{0}$ is $E^c = \{(x_1, x_2, y) : y = 0\}$, and the stable subspace at $\mathbf{0}$ is $E^s = \{(x_1, x_2, y) : x_1 = x_2 = 0\}$.

Since $u = 0$, there is no $h_2$ to compute. Since the first order approximation is the zero matrix, $h_1$ vanishes to second order, and considering the highest powers in the nonlinear terms, we see that we shall not be needing terms higher than second order, either, so we have an ansatz of the form

$$h_1(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + O(|\mathbf{x}|^3).$$

From this we have

$$\mathrm{D}h_1(x_1, x_2) = \begin{pmatrix} 2ax_1 + bx_2 \\ bx_1 + 2cx_2 \end{pmatrix} + O(|\mathbf{x}|^2).$$

Putting this into the first equation of (24), we have that

$$\begin{aligned}0 &= \mathrm{D}h_1(x_1, x_2) \cdot F(x_1, x_2, h_1(x_2, x_2)) - (-1)h_1(x_1, x_2) - G(x_1, x_2)\\ &= \begin{pmatrix} 2ax_1 + bx_2 \\ bx_1 + 2cx_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 y - x_1 x_2^2 \\ x_2 y - x_1^2 x_2 \end{pmatrix}\Bigg|_{y = h_1(x_1, x_2)}\\ &\quad + ax_1^2 + bx_1x_2 + cx_2^2 - (x_1^2 + x_2^2) + O(|\mathbf{x}|^3).\end{aligned}$$

For $\mathbf{x}$ sufficiently small, collecting like terms, we find

$$a = 1, \qquad b = 0, \qquad c = 1.$$

This means

$$h_1(x_1, x_2) = x_1^2 + x_2^2 + O(|\mathbf{x}|^3),$$

and by the local centre manifold theorem, the flow on the centre manifold is determined by

$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2, h_1(x_1, x_2)) = x_1(x_1^2 + x_2^2) - x_1 x_2^2 + O(|\mathbf{x}|^4) = x_1^3 + O(|\mathbf{x}|^4)\\ \dot{x}_2 &= F_2(x_1, x_2, h_1(x_1, x_2)) = x_2(x_1^2 + x_2^2) - x_1^2 x_2 + O(|\mathbf{x}|^4) = x_2^3 + O(|\mathbf{x}|^4).\end{aligned}$$

This is a planar system with two zero eigenvalues and geometric multiplicity of two. Turning to polar coordinates,

$$r\dot{r} = x_1^4 + x_2^4 + O(r^5) > 0$$

for sufficiently small $r$. Therefore the origin is unstable.

This contrasts with an analysis we would have done if we only took the stable subspace approximation and set $y = 0$, resulting in

$$\dot{x}_1 = -x_1 x_2^2 \dot{x}_2 = -x_1^2 x_2,$$

from which we would have incorrectly concluded that the origin is stable, again by considering polar coordinates.

## 15. Lecture XV: Limit Sets

15.1. **$\alpha$ and $\omega$-limit sets.** In departing from discussion restricted to limit points, or fixed points, of a dynamical system, and dynamics in a neighbourhood of one such, we are taking a step away from the local theory to consider more global structures of systems. To this end, we shall need to begin with some language.

First, we should like to take a more abstract view of "trajectories". So far we have been using this term loosely to refer both, to a particular solution curve $\mathbf{x} : t \mapsto \phi_t(\mathbf{x}_0)$ of a system with flow $\phi$, starting at an initial point $\mathbf{x}_0$, i.e., as a function of $t$, as well as to the set of points

$$\Gamma_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \exists t \in \mathbb{R}\ (\phi_t(\mathbf{x}_0) = \mathbf{y})\}.$$

We shall maintain this dual usage of the word "trajectory", and point out that for autonomous systems, the latter notion can be thought of as an equivalence class of solution curves modulo a time shift, as autonomous systems have trajectories that are either disjoint or coincident.

Recall that we have defined also the forward and backward orbit of a point $\mathbf{x}_0$, which we denote a little differently here:

$$\Gamma^+_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} = \phi_t(\mathbf{x}_0),\ t \geq 0\}$$
$$\Gamma^-_{\mathbf{x}_0} = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y} = \phi_t(\mathbf{x}_0),\ t \leq 0)\}.$$

Clearly we have

$$\Gamma_{\mathbf{x}_0} = \Gamma^+_{\mathbf{x}_0} \cup \Gamma^-_{\mathbf{x}_0}.$$

As we have seen, it is not unusual for trajectories to tend towards fixed points as $t \to \pm\infty$. We shall define two sets of limit points. The $\omega$-LIMIT POINTS of $\Gamma(\mathbf{x}_0)$ are points $p \in \mathbb{R}^d$ for which there is a sequence $t_n \geq 0$ of times tending to $\infty$ for which

$$\lim_{n\to\infty} \phi_{t_n}(\mathbf{x}_0) = p.$$

Similarly, the $\alpha$-LIMIT POINTS of $\Gamma(\mathbf{x}_0)$ consist of points $p \in \mathbb{R}^d$ for which there is a sequence $t_n \leq 0$ of times tending to $-\infty$ for which

$$\lim_{n\to\infty} \phi_{t_n}(\mathbf{x}_0) = p.$$



We denote these two sets of limit points by $\omega(\Gamma(\mathbf{x}_0))$ and $\alpha(\Gamma(\mathbf{x}_0))$, respectively. A time-reversal of the system interchanges $\omega$ and $\alpha$. Often, the dependence on $\mathbf{x}_0$ is implicit and suppressed.

Notice that these sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are invariant with respect to $\phi_t$. That is,

**Lemma 15.1.** *Let $p \in \mathbb{R}^d$ be an $\omega$ (resp. $\alpha$)-limit point of $\Gamma$. Then $\Gamma_p \in \omega(\Gamma)$ (resp. $\in \alpha(\Gamma)$).*

This statement is pretty much obvious from the definition of an $\omega$-limit point of $\Gamma$. We describe the main idea:

Let $\mathbf{x}_0 \in \Gamma$. Since the system is autonomous, we can take $\Gamma$ to be $\Gamma_{\mathbf{x}_0}$. There is then a sequence of times $\{t_n\}$ diverging to infinity such that $\phi_{t_n}(\mathbf{x}_0) \to p$ as $n \to \infty$. Using the semigroup property of the flow of an autonomous system, if $q = \phi_T(p) \in \Gamma_p$, then $\phi_T \circ \phi_{t_n}(\mathbf{x}_0) = \phi_{t_n+T}(\mathbf{x}_0) \to q$. Therefore $q \in \omega(\Gamma)$ also.

The proof for $\alpha(\Gamma)$ is analogous.

If $p$ is a point in $\alpha(\Gamma)$ or $\omega(\Gamma)$, then a trajectory through $p$ is konwn as a LIMIT ORBIT of $\Gamma$.

Recall that open sets on $\mathbb{R}^d$ are by definition the very sets that can be written as a union of open balls. A set is closed if it is the complement of an open set. Also by definition, $\varnothing$ is both open and closed. We say that a subset $E$ of a closed set $F$ is DENSE if the smallest closed set containing it is $F$ itself, for example $\mathbb{Q}$ is dense in $\mathbb{R}$. We say that $F$ is the CLOSURE of $E$. If $E$ is dense in $F$, then for any $x \in F$ and $\varepsilon > 0$, $B_\varepsilon(x) \cap E$ is non-empty. Otherwise $F \backslash B_\varepsilon(x)$ is another closed set that contains $E$, strictly smaller than $F$. If an orbit $\phi_t(x_0)$ were dense in a cloesd set $E$, then it comes arbitrarily close to any point in $E$.

In particular, closed sets contain their limit points.

Having in mind the definition of $\alpha$ and $\omega$ limit sets, it should be no surprise that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are closed sets, which if bounded, are then also compact. We state a theorem to this effect:

**Theorem 15.2.** *Let $\Gamma$ be a trajectory of a $C^1$-first order autonomous system. The limit sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are closed sets, and if $\Gamma$ is contained in a compact subset of $\mathbb{R}^d$, then $\alpha(\Gamma)$ and $\omega(\Gamma)$ are also compact.*

This follows from the definition of $\alpha(\Gamma)$ (resp. $\omega(\Gamma)$) in pretty much the same way as the Lemma previously. We again sketch the general idea:

Suppose $\omega(\Gamma)$ were not closed, then there is a sequence $\{\mathbf{x}_n\} \subseteq \omega(\Gamma)$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^d \backslash \omega(\Gamma)$. For each $n$, there is a sequence $\{t_k^n\}$ of times diverging to infinity such that $\phi_{t_k^n}(\mathbf{y}_0) \to \mathbf{x}_n$, where $\mathbf{y}_0 \in \Gamma$. But then the diagonal sequence $\phi_{t_n^n}(\mathbf{y}_0)$ also converges and its limit must be in $\omega(\Gamma)$ by definition. This limit has to be $\mathbf{x}$ — a contradiction.

The point of stating the foregoing theorem is that we can define two new notions:

A closed invariant set $A$ is an ATTRACTING SET if there is a neighbourhood $U$ of $A$ for which

$$p \in U \quad \Longrightarrow \quad (\forall t \geq 0,\ \phi_t(p) \in U) \wedge (\phi_t(p) \to A \text{ as } t \to \infty).$$

An ATTRACTOR is an attracting set that contains a dense orbit. This essentially makes attractors smallest possible/minimal attracting sets.

By definition, every attracting set is an $\omega$-limit set of every trajectory in a neighbourhood around it, but the converse does not hold. A simple counterexample to the converse is the fixed point of a saddle, which is an $\omega$-limit set for three trajectories (two separatrices and the fixed point) but not any other trajectory in a small enough neighbourhood.

It is also possible for a limit set simultaneously to to be the $\alpha$-limit set of some trajectories and the $\omega$-limit set of other trajectories. It is evident that a saddle point also has this characteristic. If they are only the former, they are UNSTABLE, if only the latter, STABLE, and if both, then SEMISTABLE.



15.2. **Examples.**

**Example 15.1.** Consider the system

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 + y^2)\\ \dot{y} &= x + y(1 - x^2 - y^2).\end{aligned}$$

First order analysis tells us that there is a fixed point at $(0,0)$, and that the linearized system exhibits an unstable focus. The Hartman-Grobman theorem for $C^2$-systems (or Thm. 12.2) suggests that this implies an unstable focus in a neighbourhood of $(0,0)$ for the full system.

In polar coordinates, we can see more:

$$\begin{aligned}\dot{r} &= r(1 - r^2)\\ \dot{\vartheta} &= 1.\end{aligned}$$

As expected, $\dot{\vartheta} > 0$, and $\dot{r} > 0$ for $r < 1$, indicating an unstable focus near $(0,0)$, but we see that if we start at $r = 1$, we remain on the unit circle, and outside this unit circle, $\dot{r} < 0$. Therefore, the unit circle is a limit cycle that we would not have seen from just a local analysis of the system.

**Example 15.2.** Let us identify $\mathbb{R}^2$ with $\mathbb{C}$, and consider the discrete system constructed by the flow of

$$\dot{z} = 2\pi\alpha i z.$$

at integral times. Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

This equation can be easily integrated:

$$z(t) = e^{2\pi i \alpha t} z_0.$$

Observe that because $\alpha$ is irrational, $z(n)$ is dense on $|z_0|\mathbb{T}$, where $\mathbb{T} = \{\zeta : |\zeta| = 1\}$. This is a result of a theorem from diophantine analysis known as Hurwitz's Theorem:

**Theorem 15.3** (Hurwitz's Theorem)**.** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then for any $Q \in \mathbb{N}$ there exists $q \leq Q$ and $p \in \mathbb{Z}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}qQ}.$$

This theorem is actually due to Dirichlet, but Hurwitz proved the best constant $1/\sqrt{5}$. Liouville extended this theorem to different orders of approximations of algebraic numbers according to their degree over $\mathbb{Q}$. Disregarding the best constant, which comes from approximating the golden ratio, it is not difficult to see why this theorem is true.

*Proof.* Let $[x]$ denote the largest integer smaller than of $x \in \mathbb{R}$ and set $\{x\} = x - [x]$. Given $Q$, since $\alpha$ is irrational, $\{q\alpha\}$ is never of the form $n/Q$. Imagine the interval $[0, 1]$ being divided up into $Q$ subintervals of equal length. By the pigeonhole principle, as $q$ ranges over $0, 1, 2, \ldots, Q$, at least two of the numbers $q_1\alpha$ and $q_2\alpha$ have non-integer parts that fall into the same subinterval. This means $|\{q_1\alpha\} - \{q_2\alpha\}| < 1/Q$. Putting the integers back in, there is an integer $p$ such that

$$|(q_1 - q_2)\alpha - p| < 1/Q.$$

Setting $q = q_1 - q_2$ yields the desired approximation. □

For any $\varepsilon > 0$, we can find $Q \in \mathbb{Z}$ such that $1/Q < \varepsilon$. and for any $Q$, we can find $n$ and $p \in \mathbb{Z}$ such that

$$|n\alpha - p| < 1/Q < \varepsilon.$$

So for any point with argument $\theta$, we know that $2\pi m\alpha$ we can get within $\varepsilon$ of $\theta$ by taking $m$ to be an integer multiple of $n$ above. This proves that $z(t)$ can get arbitrarily close to any point in $|z_0|C$, and hence is is dense over $C$.

By choosing a sequence of integral times $t_k = n_k$, we see that for any $z_0 \in \mathbb{T}$, $\phi_{t_n}(z_0)$ can converge to any point on $\mathbb{T}$, and so $\omega(\Gamma_{x_0}) = \mathbb{T}$. Notice that this is patently false for $\alpha \in \mathbb{Q}$.

This is an important example because this observation that such a flow controlled by an irrational number gives a dense, and indeed, uniformly distributed $\phi_n(z_0)$ is the fountainhead of much research in number theory, ergodic theory, and discrepancy theory.



**Example 15.3.** The Lorenz system was originally suggested as a model for atmospheric convection in 1963.

We briefly mention the Lorenz system, which has appeared in an exercise some weeks before:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y - xz\\ \dot{z} &= xy - \beta z,\end{aligned}$$

where $\sigma, \rho, \beta > 0$.

Where $\alpha = \sqrt{(\rho - 1)\beta}$, we have seen that the fixed points of this system are at

$$p_1 = (0, 0, 0)^\top, \qquad p_2 = (\alpha, \alpha, \rho - 1)^\top, \qquad p_3 = (-\alpha, -\alpha, \rho - 1)^\top.$$

By inspecting the eigenvalues for the linearization at $p_1$, we know that the system is stable for

$$(\sigma + 1)^2 < 4\sigma(1 - \rho),$$

and unstable otherwise.

We know that the centre manifold of two dimensional analytic systems can be characterized more or less completely, even if behaviours can differ from linear systems substantially. For appropriate values of $\sigma$, $\rho$ and $\beta$, the Lorenz system has a curiously complicated attractor unrivalled in complication by behaviours that two dimensions can support. This is the case even though one of the equations of the system is linear, and the remaining are also analytic.

The attractor $A$ of this system is made of an infinite number of banched surfaces each of which intersect. Trajectories, being of an autonomous system, do not intersect as they move along $A$,

but there are nevertheless periodic trajectories of arbitrarily large period, uncountably many non-periodic trajectories, and also trajectories that are dense in $A$. We refer to these attractors as "strange attractors".

Next we turn to a fuller discussion of limit cycles but confine ourselves to the plane.

15.3. **Limit Cycles.** CYCLES, or PERIODIC ORBITS are closed-curve solutions that are not equilibria, a definition exactly in line with our previous usage of the term. Since our systems are autonomous, if the trajectory $\Gamma$ is a cycle, there is a minimal $T$ independent of $\mathbf{x}_0 \in \Gamma$ for which $\phi_{t+T}(\mathbf{x}_0) = \phi_t(\mathbf{x}_0)$. This minimum $T$ we call the PERIOD of the $\Gamma$. For centres of linear systems, the period is constant over a family of periodic orbits. This is not so in general. We shall be spending the next few lectures looking at the behaviour of periodic solutions and limit cycles, especially in two dimensions.

A cycle can be itself unstable, stable, and asmptotically stable. To discuss these notions analogously to the way we did for fixed points, we digress briefly to mention that the distance from a point $\mathbf{x}$ to a set $E$ is defined as

$$d(\mathbf{x}, E) := \inf_{\mathbf{y}\in E} |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^d}.$$

Then we say that a cycle $\Gamma$ is STABLE if for every $\varepsilon > 0$, there is a neighbourhood $U$ of $\Gamma$ (an open set $U$ containing $\Gamma$) for which $\mathbf{x} \in U$ implies

$$d(\phi_t(\mathbf{x}), \Gamma) < \varepsilon.$$

The cycle $\Gamma$ is UNSTABLE if it is not stable. And analogous to asymptotic stability previously defined, $\Gamma$ is ASYMPTOTICALLY STABLE if it contains the $\omega$-limit set of every trajectory within a certain neighbourhood $U$ of itself. That is, $\Gamma$ is stable, and there exists a neighbourhood $U$ of $\Gamma$ such that $\mathbf{x} \in U$ implies

$$\lim_{t\to\infty} d(\phi_t(\mathbf{x}), \Gamma) = 0.$$



Closed curves that are equilibria of sorts (not an equilibrium in the sense that they sit at the intersection of nullclines!) constitute an important class of limit sets known as LIMIT CYCLES, which we have encountered in the specific context of centre-foci. These are cycles that are the $\alpha$ or $\omega$ set of some trajectories. And if the entire cycle is a limit set, then there is also the notion of semistability that can be defined. If there exists a neighbourhood $U$ of $\Gamma$ for which $\Gamma$ is the $\omega$-limit set (resp. $\alpha$-limit set) for every trajectory in/passing through $U$, then $\Gamma$ is an $\omega$-LIMIT CYCLE, or a STABLE LIMIT CYCLE (resp. $\alpha$-LIMIT CYCLE, or a UNSTABLE LIMIT CYCLE). If $\Gamma$ is an $\alpha$-limit set for one trajectory and an $\omega$-limit set for another trajectory, then we say that it is a SEMISTABLE LIMIT CYCLE.

Cycles, like points, have stable and unstable manifolds. For a cycle $\Gamma$, and $U$ a neighbourhood thereof, we define the local stable and unstable manifolds as

$$M_s(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to\infty}{\to} 0,\ \forall t \geq 0,\ \phi_t(\mathbf{x}) \in U\},$$

and

$$M_u(\Gamma) := \{\mathbf{x} \in U : d(\phi_t(\mathbf{x}), \Gamma) \overset{t\to-\infty}{\to} 0,\ \forall t \leq 0,\ \phi_t(\mathbf{x}) \in U\},$$

respectively. The global stable and unstable manifolds of a cycle are then

$$W^s(\Gamma) = \bigcup_{t\leq 0} \phi_t(M_s(\Gamma))$$

$$W^u(\Gamma) = \bigcup_{t\geq 0} \phi_t(M_u(\Gamma)).$$

This definition makes the global stable and unstable manifolds invariant under $\phi_t$. We shall see that in many ways, we can treat a periodic orbit as a critical point, even though there are important ways that we cannot so do.

15.4. **Some results in $\mathbb{R}^2$.** In $\mathbb{R}^2$, there is a fundamental result about simple closed curves (i.e., closed curves $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(t) = \gamma(T)$ implies $T = 1$ and $t = 0$, or vice versa), known as the Jordan curve theorem:

**Theorem 15.4** (Jordan curve theorem)**.** *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple cloesd curve. Then $\mathbb{R}^2 \backslash \{\gamma(t) : t \in [0,1]\}$ is a disjoint union of two open sets.*

In fact, simple closed curves are also known as Jordan curves. This might seem like a trivial result, but its proof is not easy. A hint of its depth is that we can characterize the topological dimension of a space by removing objects. One of the fundamental properties distinguishing $\mathbb{R}$ from $\mathbb{R}^d$, $d > 2$, is that by removing a point from $\mathbb{R}$, it becomes disconnected — it is the disjoint union of two open sets.

We call these two disjoint parts the "interior" and the "exterior".

It turns out that a limit cycle has to be approached maximally tangentially. We can think of a approach to a limit cycle as simliar to an approach to a point that sits at the centre of a focus, and we have the following theorem that is analogous to the second part of Thm.13.1 [(in an analytic planar system, if one trajectory spirals, then all do)]:

**Theorem 15.5.** *If one trajectory in the exterior of a limit cycle $\Gamma$ of a planar $C^1$-system has $\Gamma$ as its $\omega$-limit set (resp. $\alpha$-limit set), then every trajectory in some exterior neighbourhood of $\Gamma$ does so also. Moreover any such trajectory spirals towards $\Gamma$ as $t \to \infty$ (resp. $t \to -\infty$) in the sense that it intersects any line segment normal to $\Gamma$ at a point on a sequence of times $\{t_n\}$ diverging to infinity (resp. negative infinity).*

In polar terms, from any $\mathbf{y}$ in the interior of $\Gamma$, the trajectory $\phi_t(\mathbf{x}_0)$ satisfies

$$d(\phi_t(\mathbf{x}_0), \Gamma) \searrow 0, \qquad \arg(\phi_t(\mathbf{x}_0) - \mathbf{y}) \to \infty,$$

as $t \to \infty$.



**Example 15.4.** Recall from Dulac's theorem (Thm. 11.3) that analytic planar systems can have at most finitely many limit cycles. We shall look at a non-analytic planar system that exhibits a centre-focus:

$$\dot{x} = -y + x(x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}})$$
$$\dot{y} = x + y(x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}).$$

In polar coordinates this becomes:

$$\dot{r} = r^3 \sin(1/r), \qquad \dot{\vartheta} = 1.$$

As we can see, at $r = 1/(n\pi)$, for every $n \in \mathbb{N}$, there is a limit cycle, and between each limit cycle are alternating stable and unstable spirals that turn in the same sense.

Hamiltonian systems often have cyclic behaviour as energy passes from one mode to another and then back again, being conserved throughout the cycle. We shall consider them in greater detail in the next lecture. Recalling Example 11.1, we call the separatrices of the saddles HETEROCLINIC as they "bend towards" different limit sets. Elliptic domains would have what are known as HOMOCLINIC orbits for the same reason. It is slightly a matter of perspective whether an orbit is homo- or hetero- clinic, according as when we should like to consider different points in the same limit set.

## 16. Lecture XVI: Poincaré Map and Stability

We shall have a slight respite from the confinement to the plane in this lecture and consider again periodic orbits of autonomous systems in $\mathbb{R}^d$.

16.1. **The Poincaré Map.** A basic construction in the study of periodic orbits is the Poincaré map. Suppose $\Gamma$ is a periodic orbit of a $C^1$-first order autonomous system in a neighbourhood $U \subseteq \mathbb{R}^d$. The Poincaré map is the map $\mathbf{\Pi} : U \to U$ defined thus: Let $\mathbf{x}_0$ be a point on $\Gamma$. Let $\Sigma$ be the hyperplane perpendicular to $\Gamma$ at $\mathbf{x}_0$. That is, if $\Gamma$ is defined by $\gamma : [0,1] \to \mathbb{R}^d$ so that $\gamma(0) = \mathbf{x}_0$, then

$$\Sigma = \{\mathbf{y} \in \tilde{U} : (\mathbf{y} - \mathbf{x}_0) \cdot \gamma'(0) = 0\},$$

where $\tilde{U}$ is a small neighbourhood around $\mathbf{x}_0$ (not the entire $\Gamma$).

As $\Gamma$ is a trajectory, we know that $\gamma'(0) = \mathbf{x}'(0) = f(\mathbf{x}_0)$, where $f$ is the function defining our dynamics $\dot{\mathbf{x}} = f(\mathbf{x})$.

If $\mathbf{x}$ is a point in a small neighbourhood of $\mathbf{x}_0$, and $\mathbf{x} \in \Sigma$, then we expect $\phi_t(\mathbf{x})$ to intersect $\Sigma$ again after some time $\tau(\mathbf{x})$ at $\phi_{\tau(\mathbf{x})}(\mathbf{x})$. We define $\mathbf{\Pi} : \Sigma \to \Sigma$ as the map $\mathbf{\Pi} : \mathbf{x} \to \phi_{\tau(\mathbf{x})}(\mathbf{x})$, so that in fact $\mathbf{\Pi}$ is not defined on $U$ but on a codimension one subset $\Sigma$. We can also allow $\Sigma$ to be a smooth, curved codimension one surface transverse to $\Gamma$ at $\mathbf{x}_0$.

The fact that this time $\tau(\mathbf{x})$ and the point $\phi_{\tau(\mathbf{x})}(\mathbf{x})$ is well-defined in a neighbourhoood $U \cap \Sigma$ of $\mathbf{x}_0$ is a direct consequence of the implicit function theorem for the map $F : \mathbb{R}_{\geq 0} \times \Sigma \to \Sigma$ given by

$$F(t, \mathbf{x}) = (\phi_t(\mathbf{x}) - \mathbf{x}_0) \cdot f(\mathbf{x}_0).$$

The level set $F(t, \mathbf{x}) = 0$ gives us the points $\mathbf{x}$ in the neighbourhood of $\mathbf{x}_0$ on $\Sigma$ and their first return times, $\tau(\mathbf{x})$.

By thinking about an iterative map, we can see that if $|\nabla \mathbf{\Pi}(\mathbf{x})| < 1$ , then we have stability. In $\mathbb{R}^2$, we have the following theorem:



**Theorem 16.1.** *Let $\gamma$ be a periodic solution of period $T$ for the $C^1$- planar system $\dot{\mathbf{x}} = f(\mathbf{x})$. The derivative of the Poincaré map $\mathbf{\Pi}$ along a straight line normal to $\gamma'(0)$ is given by*

$$\mathrm{D}\mathbf{\Pi}(\gamma(0)) = \exp\Big(\int_0^T (\nabla \cdot f)(\gamma(t))\, \mathrm{d}t\Big).$$

Recall that $\mathbf{\Pi}$ is defined on a codimension one subset, so that $\mathbf{\Pi}$ and $\mathrm{D}\mathbf{\Pi}$ are both scalar objects for planar systems.

The reason this is result true is slightly complicated, and depends on Floquet's theorem, which we shall discuss later. But since $\nabla \cdot f = \mathrm{tr}(\mathrm{D}f)$, if we can write

$$(\mathrm{D}f)(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M}), \tag{25}$$

for some invertible-matrix-valued $\mathbf{M}$, we can use Jacobi's formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log(\det(\mathbf{M}))) = \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M}))$$

to get that

$$\exp\left(\int_0^T \mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t} \log(\mathbf{M}))\, \mathrm{d}t\right) = \frac{\det(\mathbf{M}(T))}{\det(\mathbf{M}(0))} = \det(\mathbf{M}(T)).$$

Notice from (25), by normalizing $\mathrm{D}f(\gamma(0))$, we have

$$\mathbf{M}(t) = \exp\left(\int_0^t \mathrm{D}f(\gamma(s))\, \mathrm{d}s\right). \tag{26}$$

This can be compared to (27) below.

From the above, we see that the periodic solution is a stable or unstable limit cycle according as

$$\int_0^T (\nabla \cdot f)(\gamma(t))\, \mathrm{d}t$$

is negative or positive. If this quantity is zero, then $\gamma$ belongs to a continuous band of cycles all with the same period.

We shall see that the derivative of the Poincaré map, $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0)$, for $\mathbf{x}_0$ such that $\Gamma_{\mathbf{x}_0}$ is periodic, serves very much "near" a periodic orbit, as the linearization $\mathrm{D}f$ does around a critical point. This shall be the basis for a stable manifold theorem for periodic orbits.

16.2. **Stable Manifold Theorem for Periodic Orbits.** In order to achieve higer-dimensional results analogous to the one above in which the stability of a limit cycle is seen to depend on $\mathrm{D}\boldsymbol{\Pi}$, with a Poincaré map $\boldsymbol{\Pi}$ defined along some codimension one subset $\Sigma$ transverse to the limit cycle at a point $\mathbf{x}_0$, we shall attempt to recreate a first-order analysis similar to our first-order analysis about critical points.

The LINEARIZATION about a periodic orbit $\Gamma = \{\gamma(t) : t \in [0, T]\}$ of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = f(\mathbf{x}(t))$$

is defined as the *nonautonomous* system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathrm{D}f(\gamma(t))\mathbf{x}(t).$$

The fundamental matrix solutions for the linearized nonautonomous system is a matrix valued $C^1$ function $\Phi$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = \mathrm{D}f(\gamma(t))\Phi,$$

completely analogous to the definition in (12) of Lecture 3. And for the same reason, as there are no non-homogeneous terms, we can write

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0.$$

Arguing especially heuristically, using integrating factors, by normalizing $\mathrm{D}f(\gamma(0))$, we have that

$$\mathbf{x}(t) = \exp\left(\int_0^t \mathrm{D}f(\gamma(s))\, \mathrm{d}s\right)\mathbf{x}_0. \tag{27}$$

We can compare this with (26) above and see that the eigenvalues of $\mathbf{M}(T)$ tells us exactly how the Poincaré map acts in a small enough neighbourhood of $\gamma(0)$ if the cycle is "hyperbolic". We shall clarify the meaning of this caveat in the main theorem of this lecture.

Recall that for autonomous linear systems, we can choose $\Phi(t) = \exp(\mathbf{A}t)$, the flow. Floquet's theorem gives us a way to write $\Phi$ in (27) in a more similar way in the nonautonomous case:

**Theorem 16.2** (Floquet's Theorem)**.** *Suppose $\mathrm{D}f(\gamma(t))$ is a continuous, matrix-valued, periodic function with period $T$. Then for all $t \in \mathbb{R}$, we can write the fundamental matrix solution defined above in the form*

$$\Phi(t) = \mathbf{Q}(t)\exp(\mathbf{B}t), \tag{28}$$

*where $\mathbf{Q}(t)$ is a non-singular, matrix-valued, $T$-periodic function and $\mathbf{B}$ is a constant matrix.*

The proof of this theorem turns on the simple observation that if a matrix $\mathbf{C}$ is non-singular, then we can define its logarithm — that is, find a possibly complex matrix $\mathbf{B}$ such that $\mathbf{C} = \exp(\mathbf{B}T)$. The matrix $\mathbf{C}$ in this case comes directly from the representation of the solution by the fundamental matrix solution, $\mathbf{C} := \Phi^{-1}(0)\Phi(T)$. Then it holds that the remaining factor $\mathbf{Q}(t)$ must be

$$\mathbf{Q}(t) := \Phi(t)\exp(-\mathbf{B}t) = \Phi(t)\Phi^{-1}(T)\Phi(0).$$

It may be checked readily that $\mathbf{Q}$ is periodic with period $T$, to see that it is non-singular one has but to notice that if $\Phi$ takes the form (26), $\mathbf{Q}$ is an exponentiation.

Written out thus, Floquet's theorem seems like a mere technical result, but the consequences of this representation (28) is far-reaching. Using the change-of-variables $\mathbf{y} = \mathbf{Q}^{-1}(t)\mathbf{x}$, a direct calculation shows that the nonautonomous linearized system can be written as the autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = \mathbf{B}\mathbf{y}.$$

Now we can bring the full suite of linear methods developed thus far to bear.

We know that $\mathbf{Q}$ is non-singular and periodic (and if everything is smooth, also bounded), therefore the stability of the cycle $\Gamma$ is determined by the eigenvalues of $\mathbf{B}$ — these eigenvalues are called the CHARACTERISTIC EXPONENTS of $\gamma(t)$, and are determined modulo $2\pi i$. Trajectories for which the characteristic exponents all have non-zer real parts are known as HYPERBOLIC PERIODIC ORBITS.

The eigenvalues of $\exp(\mathbf{B}T)$ are known as the CHARACTERISTIC MULTIPLIERS — these determine how far $\mathbf{x}$ moves every time it comes around to the surface $\Sigma$ again, i.e., the magnitude of $\mathbf{\Pi}(\mathbf{x}) - \mathbf{x}$, or $\mathrm{D}\mathbf{\Pi}$. We shall not write this out as a theorem, but from our deductions foregoing, it may be guessed that the characteristic multipliers are the eigenvalues of $\mathrm{D}\mathbf{\Pi}$ (if one considered $\mathbf{\Pi}$ as a map $\Sigma \subseteq U \to U$ instead of $\Sigma \to \Sigma$.

Of course, $\mathbf{B}$ has to be singular and have one zero eigenvalue because $\mathbf{\Pi}(\mathbf{x})$ does not move $\mathbf{x}$ in the direction of $\gamma'(0)$, so the multiplier in that direction must be 1, whose logarithm is 0.

These observations leads us right to the main theorem of this lecture:

**Theorem 16.3** (Stable Manifold Theorem for Periodic Orbits)**.** *Let $\Gamma = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \gamma(t)\}$ be a periodic orbit with period $T$, of a $C^1$ first order autonomous system with flow $\phi_t$. Suppose $\gamma(t) = \phi_t(\mathbf{x}_0)$. If $k$ characteristic exponent of $\gamma(t)$ have negative real parts, and $d - k - 1$ have positive real parts, then there is a $\delta > 0$ such that the stable manifold,*

$$M_s(\Gamma) = \{\mathbf{x} \in B_\delta(\Gamma) : d(\phi_t(\mathbf{x}), \Gamma) \to 0 \text{ as } t \to \infty,\ \forall t \geq 0\,(\phi_t(\mathbf{x}) \in B_\delta(\Gamma))\},$$

*is a $(k+1)$-dimensional differentiable manifold which is positively $\phi_t$-invariant, and the unstable manifold,*

$$M_u(\Gamma) = \{\mathbf{x} \in B_\delta(\Gamma) : d(\phi_t(\mathbf{x}), \Gamma) \to 0 \text{ as } t \to -\infty,\ \forall t \geq 0\,(\phi_t(\mathbf{x}) \in B_\delta(\Gamma))\},$$

*is a $(d - k - 1)$ dimensional differentiable manifold which is negatively $\phi_t$-invariant. Furthermore, the stable and unstable manifolds intersect transversally.*

Subanifolds $M \subseteq X$ and $N \subseteq X$ intersect TRANSVERSELY (or TRANSVERSALLY) if at every point $p$ of their intersection , $T_pM \oplus T_pN = T_pX$. This transversality condition is analogous to the condition of tangency to invariant subspaces in the stable manifold theorem for critical points.

In fact, we can say a little more, though we shall not seek to justify this:

**Theorem 16.4.** *Under the hypotheses of the previous theorem, the magnitude of the real parts of the characteristic exponents of the $T$-periodic orbit $\Gamma$ are all lower-bounded by $\alpha > 0$. There exists a $K > 0$ such that for each $\mathbf{x} \in M_s(\Gamma)$, there exists an asymptotic phase $t_0$ such that for all $t \geq 0$,*

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{-\alpha t/T},$$

*and for each $\mathbf{x} \in M_u(\Gamma)$, there exists an asymptotic phase $t_0$ such that for all $t \geq 0$,*

$$|\phi_t(\mathbf{x}) - \gamma(t - t_0)| \leq Ke^{\alpha t/T}.$$

That is, not only do orbits approach a limit cycle $\Gamma$, they also become phase-locked with $\Gamma$. In a way this is not too surprising, because the flux $f$ governing the dynamics is continuous. These are essentially Gronwall-type estimates.

As with the stable manifold theorem for critical points, there is an associated (weak) centre manifold theorem, asserting the existence of a centre manifold of dimension equal to one less than the number of characteristic exponents with zero real parts. But we should like to revisit the condition of Thm. 16.1,which chararcterised stability with a calculable quantity. This can be generalized to higher dimsions for reasons we have already touched upon in our deductions leading up to the stable manifold theorem for periodic orbits.

**Theorem 16.5.** *Let $\gamma$ be a $T$-periodic orbit of a $C^1$-first order autonomous system with flux $f$. A necessary but not generally sufficient condition for the orbit $\gamma$ to be asymptotically stable is that*

$$\int_0^T (\nabla \cdot f)(\gamma(t))\, \mathrm{d}t \leq 0.$$

The non-sufficiency comes from the fact that we need the eigenvalues of $\int_0^T \mathrm{D}f(\gamma(t))\, \mathrm{d}t$ all to be negative except the 0 eigenvalue arising from the fact that the Poincaré map maps onto a codimension one surface, whereas the condition stated in the theorem is merely the trace of this quantity. This would have been enough in dimension $d = 2$, where only one eigenvalue is unspecified apart from the 0.

## 17. Lecture XVII: Poincaré-Bendixson Theorem

Let us start off with a lemma on general $\omega$-limit sets:

**Lemma 17.1.** *Let $\omega(\Gamma)$ be a bounded limit set. Then it is connected.*

We say that a set $X$ is NOT CONNECTED if there are disjoint open sets $U$ and $V$ each of which intersects $X$, and $X \subseteq U \cup V$. A set is CONNECTED if it is not disconnected.

*Proof.* Let $\Gamma = \Gamma_{\mathbf{x}_0}$ and $\mathbf{x}(0) = \mathbf{x}_0$.

Suppose $\omega(\Gamma)$ is not connected. Then it is the union of two disjoint closed sets $A$ and $B$ because by Thm.15.2, $\omega(\Gamma)$ is closed. Since $A$ and $B$ are closed, there is a positive distance between them:

$$d(A,B) = \inf_{\mathbf{x}\in A, \mathbf{y}\in B} |\mathbf{x}-\mathbf{y}|_{\mathbb{R}^d} > \delta > 0.$$

Since $A$ and $B$ are in $\omega(\Gamma)$, there are aritrarily large times for which $d(\mathbf{x}(t), A) < \delta/2$, and also arbitrarily large times for which $d(\mathbf{x}(t), A) > \delta/2$. Since $t \mapsto \mathbf{x}(t)$ is continuous, so is $t \mapsto d(\mathbf{x}(t), A)$. Therefore by the intermediate value theorem, there is a sequence of times $t_n$ for which $d(\mathbf{x}(t_n), A) \to \delta/2$. Since $A$ and $B$ are bounded, so is the set of points $\{\mathbf{x}(t_n)\}$. Therefore there is a subsequence such that $\mathbf{x}(t_{n_k})$ converges to a point $\mathbf{z}$, with $d(\mathbf{z}, A) = \delta/2$.

But by the triangle inequality, $\mathbf{z}$ is neither in $A$ nor in $B$ — a contradiction. □

Cycles of finite period are always their own $\alpha$- and $\omega$- limit sets. Last time we considered the stability of cycles with finite period $T$. This also covered any particular cycle in an attractor with cycles of arbitrarily large periods. This time we cast our minds back to a brief discussion on separatrices, homoclines, and heteroclines, to discuss cycles that contain critical points. These are topologically, cycles, and they are often rectifiable — they are simple closed curves — but they have infinite period.

We shall call a SEPARATRIX CYCLE of a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$ a continuous image of a circle which is a finite union of critical points and compatibly oriented separatrices connecting them. That is, a union of points $\mathbf{x}_i$ and separatrices $\Gamma_i$ such that $\alpha(\Gamma_i) = \mathbf{x}_i$, and $\omega(\Gamma_i) = \mathbf{x}_{i+1}$, for $i = 1, \ldots, m$, and $\mathbf{x}_{m+1} = \mathbf{x}_1$. A GRAPHIC is a finite union of compatibly oriented separatrix cycles.

In this lecture we return to the plane and our goal here is singular: the proof of the theorem following.



**Theorem 17.2** (Poincaré-Bendixson Theorem)**.** *Suppose the forward orbit $\Gamma^+$ of a trajectory $\Gamma$ of a $C^1$-planar system is contained in a compact subset $F \subset \mathbb{R}^2$. Either $\omega(\Gamma)$ is a periodic orbit or it contains a critical point.*

It also happens that for analytic systems we can be slightly more precise, as we have seen is often the case, and the alternative provided for by the theorem for which $\omega(\Gamma)$ contains a critical point can be refined further to say that $\omega(\Gamma)$ *is* a critical point, or $\omega(\Gamma)$ is a graphic.

In order to prove the theorem we shall be needing auxiliary results. We say that a closed, (straight) line segment $\ell$ is a TRANSVERSAL to a system if it is transverse as a manifold to any trajectory that intersects it. In particular, it does not contain a critical point. Recall that transversality means that if the trajectory $\Gamma$ and the line segment $\ell$ intersect at an interior point $\mathbf{x}_0$, then $T_{\mathbf{x}_0}\Gamma \otimes T_{\mathbf{x}_0}\ell = \mathbb{R}^2$. In simpler language, $\Gamma$ and $\ell$ cannot be tangent at $\mathbf{x}_0$. Since the derivative of $\Gamma$ at $\mathbf{x}_0$ is $f(\mathbf{x}_0)$, this also means that the derivative of $\ell$ at $\mathbf{x}_0$ is not some non-zero scalar multiple of $f(\mathbf{x}_0)$.

We say that $\mathbf{x}_0$ is a regular point of the system unless it is a critical point. Unless $\mathbf{x}_0$ is a critical point, $\Gamma$ continues beyond $\mathbf{x}_0$, and therefore $\Gamma$ crosses $\ell$ at any interior point $\mathbf{x}_0$ that is a REGULAR POINT. Our first lemma relaxes this observation slightly:

**Lemma 17.3.** *Let $\mathbf{y}_0$ be an interior point of a transversal $\ell$ and regular point of a $C^1$-autonomous first order system. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that every trajectory passing through a point in $B_\delta(\mathbf{y}_0)$ crosses $\ell$ at some time $|t| < \varepsilon$.*

That is, if $\ell$ is transverse to the system at a regular point $\mathbf{y}_0$, then it is transverse to the system along a small neighbourhood $B_\delta(\mathbf{y}_0) \cap \ell$ of points.

*Proof.* Since limit sets are cloesd by Thm. 15.2, we know that in a small enough neighbourhood $B_\delta(\mathbf{y}_0)$ of a regular point, all points are regular. By assumption, $\ell$ is a transveral, so it is transverse to any trajectory that intersects it, and we only have to prove that any trajectory passing through a small enough neighbourhood intersects $\ell$.

By the continuity of $f$ and boundedness of $\nabla f$, and the regularity of $\mathbf{y}_0$, we know the trajectory $\Gamma_{\mathbf{y}}$ through any point $\mathbf{y} \in B_\delta(\mathbf{y}_0) \cap \ell$ must be $f(\mathbf{y}) = f(\mathbf{y}_0) + \nabla f(\mathbf{z}) \cdot (\mathbf{y} - \mathbf{y}_0) \approx f(\mathbf{y}_0)$ for $\mathbf{z} \in B_\delta(\mathbf{y}_0)$ in a small neighbourhood of the non-zero vector $f(\mathbf{y}_0)$. By the continuity of these trajectories, we can find a small enough neighbourhood such that every trajectory passing through that neighbourhood intersects $\ell$.

□

**Lemma 17.4.** *Let $\mathbf{x}_0$ be a point on a positively invariant set, and let $\ell$ be a transversal. Then $\Gamma^+(\mathbf{x}_0)$ intersects $\ell$ in a monotone sequence (that is, if $\mathbf{x}_i$ is the $i$th intersection, then $\mathbf{x}_i$ lies between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ on $\ell$) at only finitely many points.*

*Proof.* Let $\{t_i\}$ be the increasing sequence of time for which $\phi_{t_i}(\mathbf{x}_0) = \mathbf{x}_i$.

By the previous theorem, the region enclosed by the Jordan curve formed by the segment of $\ell$ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, and the arc $\{\phi_t(\mathbf{x}_0) : t_{i-1} \le t \le t_i\}$ is positively invariant. Therefore $\mathbf{x}_{i+1}$ must be in this region, and hence $\mathbf{x}_{i+1}$ cannot be on the segment of $\ell$ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$.

Since $\ell$ is closed and bounded, if $\{\mathbf{x}_i\}$ are an infinite sequence, then $\mathbf{x}_i$ converges on $\ell$, and this limit must be a limit point, which is excluded by construction.

□

Another way to see this is that if $\mathbf{x}_{i+1}$ were between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, then $\ell$ cannot be a transversal — it must either be tangent to a trajectory at some point between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, or contain a critical point there, by Rolle's theorem/the intermediate value theorem.



**Lemma 17.5.** *Let $\ell$ be a transversal to the system for which $\omega(\Gamma)$ is a limit set. Then $\omega(\Gamma)$ intersects $\ell$ at at most one point.*

*Proof.* Let $\Gamma$ be the image of the solution curve $t \mapsto \mathbf{x}(t)$ over $\mathbb{R}$.

Suppose $\omega(\Gamma)$ intersects $\ell$ at two points, $\mathbf{y}_1$ and $\mathbf{y}_2$. Then there is a sequence of times $\{t_n\}$ and a sequence of time $\{s_n\}$ each tending to infinity for which $\mathbf{x}(t_n) \to \mathbf{y}_1$ and $\mathbf{x}(s_n) \to \mathbf{y}_2$. Since any neighbourhood of $\mathbf{y}_i$ intersects $\ell$, and each time sequence tends to infinity, by the Lemma 17.3 we can choose $\mathbf{x}(t_n)$ and $\mathbf{x}(s_n)$ to lie on $\ell$. However, this violates the monotonicity established in the lemma immediately foregoing.

□

**Lemma 17.6.** *If $\Gamma$ and $\omega(\Gamma)$ share a point, then $\Gamma$ is either a critical point or a periodic orbit.*

*Proof.* Suppose $\mathbf{x}_0 \in \Gamma \cap \omega(\Gamma)$. Then there is a solution $\mathbf{x}(t)$ for which $\Gamma = \{\mathbf{x}(t) : t \in \mathbb{R}\}$ and a sequence of times $t_n$ with $n = 1, 2, \ldots$, and $t_n \to \infty$ such that $\mathbf{x}_n := \mathbf{x}(t_n) \in \Gamma$ and $\mathbf{x}_n \to \mathbf{x}_0$. As $\mathbf{x}_0 \in \Gamma$, there is a (finite) $T \in \mathbb{R}$ such that $\mathbf{x}_0 = \mathbf{x}(T)$.

Since $|\mathbf{x}(t_n) - \mathbf{x}(T)| \to 0$, by continuity, we know that there are points $t_n'$ such that $\mathbf{x}(t_n') = x(T)$ for sufficiently large $n$, and $|t_n - t_n'| \to 0$. This is the intermediate value theorem. This implies that $t \mapsto \mathbf{x}(t)$ is periodic. A critical point is a periodic orbit with period zero. □

**Lemma 17.7.** *If $\omega(\Gamma)$ contains no critical point but contains a periodic orbit $\Gamma_0$, then $\omega(\Gamma) = \Gamma_0$.*

*Proof.* Suppose $\Gamma_0 \subset \omega(\Gamma)$. Since $\omega(\Gamma)$ is connected, $A = \omega(\Gamma) \backslash \Gamma_0$ is not closed, and $\Gamma_0$ contains some limit point $\mathbf{y}_0$ of $A$. That means that every neighbourhood $B_\delta(\mathbf{y}_0)$ intersects $A$.

Let $\ell$ be a transversal of the system through $\mathbf{y}_0$. For a small enough neighbourhood $B_\delta(\mathbf{y}_0)$, every trajectory passing through $B_\delta(\mathbf{y}_0)$ also crosses $\ell$. Suppose $\mathbf{y} \in B_\delta(\mathbf{y}_0) \cap A$ and $\Gamma_{\mathbf{y}}$ crosses $\ell$ at $\mathbf{y}_1$.

Since $\Gamma_{\mathbf{y}} \subseteq A \subseteq \omega(\Gamma)$, it must that $\Gamma_{\mathbf{y}}$ is a limit orbit distinct from $\Gamma_0$. Therefore $\mathbf{y}_0 \neq \mathbf{y}_1$ and $\omega(\Gamma)$ crosses a transversal at two distinct points — a contradiction.

□

*Proof of Thm.17.2.* Suppose $\omega(\Gamma)$ does not contain a critical point.

If $\Gamma$ is a periodic orbit, then $\Gamma \subseteq \omega(\Gamma)$, then in fact $\Gamma = \omega(\Gamma)$.

If $\Gamma$ is not a periodic orbit, and $\omega(\Gamma)$ only contains regular points, then choosing $\mathbf{y} \in \omega(\Gamma)$ and flowing it shows that there is a limit orbit $\Gamma_0 \subseteq \omega(\Gamma)$. We shall show that $\Gamma_0$ is periodic, from which we shall conclude that $\omega(\Gamma) = \Gamma_0$.

Since $\Gamma^+$ is contained in a compact set $F$, possibly by enlarging $F$ slightly, $\Gamma_0$ is also contained with a compact set.

Since $\omega(\Gamma)$ is closed, $\Gamma_0$ has an $\omega$-limit point in $\omega(\Gamma)$. Let $\ell$ be a transversal through $\mathbf{y}_0$ (i.e., $\mathbf{y}_0$ is an interior point of $\ell$). By Lemma 17.3, $\Gamma_0$ must intersect $\ell$ at some point. Since there are finitely many intersections by lemma 17.4, and $\mathbf{y}_0$ is a limit point, it must be that $\Gamma_0$ intersects $\ell$ at $\mathbf{y}_0$.

Since $\mathbf{y}_0$ is an $\omega$-limit point of $\Gamma_0$, it holds that $\mathbf{y}_0 \in \Gamma_0 \cap \omega(\Gamma_0)$, and by Lemma 17.6 and assumptions on $\omega(\Gamma)$, $\Gamma_0$ is periodic.

Finally, by Lemma 17.7, $\Gamma_0 \subseteq \omega(\Gamma)$ implies that $\omega(\Gamma) = \Gamma_0$.

□

Having attained our goal it also bears remarking on a simple observation as to when periodic orbits can be excluded, again named after Dulac:

**Theorem 17.8** (Dulac's Theorem)**.** *Let*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = f(\mathbf{x})$$

*be a $C^1$ planar system on the open set $U \subseteq \mathbb{R}^2$. If there exists a $C^1$ function $V : U \to \mathbb{R}$ such that $\nabla \cdot (Vf) \geq 0$ and not identically zero on $U$, then there are no periodic orbits lying entirely within $U$.*

This is a result of the divergence theorem:

*Proof.* Suppose that $\Gamma \subseteq U$ is a periodic orbit enclosing the set $E$. By Green's/Stoke's/divergence theorem,

$$\int_E \nabla \cdot (Vf)\, \mathrm{d}\mathbf{x} = \int_\Gamma Vf \cdot \mathrm{d}\mathbf{s},$$

where $\mathbf{s}$ is aligned to the outer normal of $E$. Since $f$ is tangent to the orbit all along $\Gamma$, the integral on the right is nought. Therefore $Vf$ cannot be non-negative (or non-positive) without being identically zero.

□

## 18. Lecture XVIII: Perturbation Theory I

Here we shall discuss a very practical tool in the analysis of dynamical systems. We shall step momentarily away from phase space and consider perturbative methods. In a way, perturbation methods is another facet of the same technique which in calculus is called "Taylor's expansion", and which we have repeatedly exploited to first order in another way.

Many physical systems have a characteristic scale or characteristic length, whether it be the diameter of a pipe in fluid flow, or the resonance frequency of beam in a structure under dynamic loading, or the frequency of the rotation of the earth in weather modelling, or something more apparently fundamental, like the Planck length, or the bond length of the hydrogen molecule.

Given a system $\dot{\mathbf{x}} = f(\mathbf{x}, \varepsilon)$, with a characteristic scale $\varepsilon$, we should like to propose an ansatz to bounded oscillatory solution of a form similar to

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \varepsilon^2 \mathbf{x}_2(t) + \cdots .$$

Inserting this into the equation we find that

$$\dot{\mathbf{x}_0}(t) + \varepsilon \dot{\mathbf{x}_1}(t) + \cdots = f(\mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + \cdots, \varepsilon)$$

Next we can collect the terms of the same order in $\varepsilon$, and what is hoped is that we should arrive at a collection of equations at each order that are simple not too coupled, each with suitably bounded solutions, so that $\mathbf{x}(t)$ can be well-approximated by short truncated expansion for small $\varepsilon$. We shall discover that there are many ways to do this badly.

### 18.1. **Examples.** We being with a few examples.

**Example 18.1.** First we consider a simple 1-dimensional equation with a small linear part, away from its fixed point:

$$\dot{x} = -x + \varepsilon x^2, \qquad x(0) = 1.$$

Writing $x(t) = x_0(t) + \varepsilon x_1(t) + \cdots$, we find

$$\begin{aligned}
&\dot{x}_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3) \\
&= -x_0(t) - \varepsilon x_1(t) - \varepsilon^2 x_2(t) + \varepsilon (x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t))^2 + O(\varepsilon^3) \\
&= -x_0(t) + \varepsilon(-x_1(t) + x_0^2(t)) + \varepsilon^2 (x_2(t) + 2x_0(t)x_1(t)) + O(\varepsilon^3).
\end{aligned}$$



Collecting the terms we find that if the equation above is to hold for every sufficiently small $\varepsilon > 0$, then

$$\begin{aligned}
\dot{x}_0(t) &= -x_0(t), \\
\dot{x}_1(t) &= -x_1(t) + x_0^2(t), \\
\dot{x}_2(t) &= -x_2(t) + 2x_0(t)x_1(t).
\end{aligned}$$

Since $x(0) = 1$ so that for every $\varepsilon > 0$, $1 = x_0(0) + \varepsilon x_1(0) + \ldots$, we find (by matching terms again), that $x_0(0) = 1$ and $x_n(0) = 0$ for $n > 0$.

These equations can be solved in sequence, which is not the general situation. They imply that

$$\begin{aligned}
x_0(t) &= e^{-t}, \\
x_1(t) &= e^{-t} - e^{-2t}, \\
x_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}.
\end{aligned}$$

We can in fact solve the full equation explicitly for comparison:

$$\begin{aligned}
t &= \int_0^{x(t)} \frac{1}{-y(1-\varepsilon y)} \, \mathrm{d}y \\
&= -\int_1^{x(t)} \frac{1}{y} + \frac{\varepsilon}{1-\varepsilon y} \, \mathrm{d}y = -\log(x(t)) + \varepsilon \log(1 - \varepsilon x(t)) - \varepsilon \log(1-\varepsilon),
\end{aligned}$$

so

$$x(t) = \frac{e^{-t}}{1-\varepsilon+\varepsilon e^{-t}} = e^{-t}\sum_{n=0}^{\infty}\varepsilon^n(1-e^{-t})^n.$$

This verifies our pertubative method applied to this problem.

It is, however, not often the case that the perturbative expansion around $\varepsilon = 0$ (and, essentially, $t = 0$) holds well for all time $t > 0$.

**Example 18.2.** Let us consider another classical example where the function itself does not decay in time but oscillates indefinitely:

$$\ddot{x} + \varepsilon x + x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$

The solution is easily found to be $x(t) = \cos(\sqrt{1+\varepsilon}t)$.

Proceeding with the ansatz

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots,$$

our perturbative method yields the following set of equations:

$$\begin{aligned}\ddot{x}_0(t) &= -x_0(t),\\ \ddot{x}_1(t) &= -x_0(t) - x_1(t),\\ \ddot{x}_2(t) &= -x_1(t) - x_2(t),\end{aligned}$$

to second order. The corresponding initial conditions are:

$$x_0(0) = 1, \qquad 0 = \dot{x}_0(0) = x_1(0) = \dot{x}_1(0) = \cdots.$$

These yield:

$$x_0(t) = \cos(t), \quad x_1(t) = -\frac{1}{2}t\sin(t), \quad x_2(t) = \frac{1}{8}(t\sin(t) - t^2\cos(t)). \tag{29}$$

The asymptotic expansion given by $x(t) = x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)$ is correct up to $O(\varepsilon^3)$ for finite times $t \in [0, T]$, $\varepsilon T \ll 1$, but if we allow $t \to \infty$, the higher order terms dominate the zeroth order term for $\varepsilon t \approx 1$.



**Example 18.3.** Finally let us consider an example in which the exact solution does decay to zero and yet a naive perturbative method yields a quickly diverging solution at any fixed order as $t \to \infty$:

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$

As can be verified, the exact solution is

$$x(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}}e^{-\varepsilon t/2}\sin(\sqrt{1-\varepsilon^2/4}t).$$

Attempting a naive perturbative ansatz, we find to first order that:

$$\begin{aligned}0 &= \ddot{x}_0(t) + x_0(t),\\ 0 &= \ddot{x}_1(t) + \dot{x}_0(t) + x_1(t).\end{aligned}$$

The corresponding initial conditions are:

$$\dot{x}_0(0) = 1, \qquad 0 = x_0(0) = x_1(0) = \dot{x}_1(0) = \cdots.$$

These yield:

$$x_0(t) = \cos(t), \quad x_1(t) = -\frac{1}{2}t\sin(t),$$

and we again encounter the same problem of SECULAR TERMS — higher and higher order terms that come to dominate the expansion as $t \to \infty$.

We shall now consider a method to rectify our expansions. Observe that the difficulty here is primarily that the ocillatory functions $x_k$ in the perturbative expansion do not quite match the exact solution in frequency. Over time there must needs be beating behaviour unless there are those polynomial factors present to cancel them out in what prima facie looks to be a rather divergent manner (we have not considered how the coefficients $\{-1/2, 1/8, \ldots\}$ decay), and in any case. we shall be needing far too many terms much too quickly as $t \to \infty$ to understand the behaviour of the solution as $t$ grows.

18.2. **Poincaré-Lindstedt method.** Having made our observation as we have done, we see that one simple way to rectify this is to scale time properly as well so as to match the exact solution.

Let us introduce then a scaled time variable:

$$\tau(t, \varepsilon) = \omega(\varepsilon)t = (1 + \omega_1\varepsilon + \omega_2\varepsilon 2 + \ldots)t,$$

and consider the time-scaled ansatz

$$x(t, \varepsilon) = x_0(\tau(t, \varepsilon)) + \varepsilon x_1(\tau(t, \varepsilon)) + \varepsilon^2 x_2(\tau(t, x\varepsilon)) + \cdots .$$

This consideration is the basis of the Poincaré-Lindstedt method.

Applying this expansion to Example 18.2, we can insert the ansatz into

$$\ddot{x} + (1 + \varepsilon)x = 0.$$

Now

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x \circ \tau)(t) = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x(\tau)\dot{\tau}^2 + \frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\ddot{\tau} = \ddot{x}(\tau)(1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \ldots)^2 + 0.$$

And so we obtain, to first order,

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_0(\tau) = -x_0(\tau),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_1(\tau) = -2\omega_1\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}x_0(\tau) - x_0(\tau) - x_1(\tau) = -x_1(\tau) + (2\omega_1 - 1)x_0(\tau).$$

The attendent intial conditions being unchanged from Example 18.2, we find that

$$x_0(\tau) = \cos(\tau).$$

Now observe that by choosing $\omega_1 = 1/2$, we have $x_1(t) \equiv 0$ since $x_1(0) = \dot{x_1}(0) = 0$.

Therefore to first order, we expect our solution to be

$$x(t) = \cos((1 + \varepsilon/2)t) + O(\varepsilon^2).$$

This is very much better an expansion than (29) — at least the first order is bounded in $t$. Moreover, on comparison with the exact solution, we see that we actually got the Taylor expansion around $t = 0$ of the correct frequency.

18.3. **Multi-scale expansion.** To secure a similar correction for Example 18.3, we shall require a generalization of the Poincaré-Lindstedt method.

In Example 18.3, we considered the Cauchy problem

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1,$$

which has the exact solution

$$x(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin(\sqrt{1-\varepsilon^2/4}\, t).$$

Notice that there is a slowly decaying factor that happens on the order of $\varepsilon t \approx 1$ and another quicker, oscillatory behaviour on the time scale $t \approx 1$. In particular, the slow scale dynamics is not oscillatory. This suggests that perhaps we can decouple the dynamics acting on a slow time scale with the dynamics acting over a quick time scale. Therefore we introduce the variables

$$\tau_1 = t, \qquad \tau_2 = \varepsilon^\alpha t,$$

for some $\alpha > 0$.

We then postulate an ansatz of the form

$$x(t) = x(\tau_1, \tau_2) = x_0(\tau_1, \tau_2) + \varepsilon x_1(\tau_1, \tau_2) + \varepsilon^2 x_2(\tau_1, \tau_2) + \cdots .$$

Putting the ansatz into the equation, we find that

$$(\partial_{\tau_1}^2 + 2\varepsilon^\alpha \partial_{\tau_1}\partial\tau_2 + \varepsilon^{2\alpha}\partial_{\tau_2}^2)x(t) + \varepsilon(\partial_{\tau_1} + \varepsilon^\alpha \partial_{\tau_2})x(t) + x(t) = 0.$$

Having two times, the inital conditions no longer guarantee a unique solution. We shall use the freedom thus afforded to elimate secular terms.

Again we can sort the terms by powers of $\varepsilon$. To zeroth order we have

$$(\partial_{\tau_1}^2 + 1)x_0 = 0, \qquad x_0(0,0) = 1, \partial_{\tau_1} x_0(0,0) = 1.$$

The general solution is

$$x_0(\tau_1, \tau_2) = A(\tau_2)\sin(\tau_1) + B(\tau_2)\cos(\tau_1), \qquad A(0) = 1, \quad B(0) = 0. \tag{30}$$

To $\varepsilon$ order, we have so far

$$\partial_{\tau_1}^2 x_1 + \partial_{\tau_1} x_0 + x_1 = 0.$$

The first order term imposes a secular term, and so we should like to balance this growth in $\tau_1$ by growth in the opposite direction with $\tau_2$. The only term available to us if $\alpha > 0$ is

$$2\varepsilon^\alpha \partial_{\tau_1}\partial_{\tau_2} x_0.$$

This forces us to choose $\alpha = 1$. And the first order equation is then modified to

$$(\partial_{\tau_1}^2 + 1)x_1 = -2\partial_{\tau_1}\partial_{\tau_2} x_0 - \partial_{\tau_1} x_0, \qquad x_1(0,0) = 0, \quad \partial_{\tau_1} x_1(0,0) = -\partial_{\tau_2} x_0(0,0).$$

This equation is not too difficult to solve when we substitute $x_0$ from above into it:

$$(\partial_{\tau_1}^2 + 1)x_1 = (2\dot{B} + B)\sin(\tau_1) - (2\dot{A} + A)\cos(\tau_1),$$

which yields

$$x_1(\tau_1, \tau_2) = C(\tau_2)\sin(\tau_1) + D(\tau_2)\cos(\tau_1) - \frac{1}{2}(2\dot{B} + B)\tau_1\cos(\tau_1) - \frac{1}{2}(2\dot{A} + A)\tau_1\sin(\tau_1),$$

with

$$C(0) = \dot{A}(0), \qquad D(0) = 0.$$

We can eliminate the secular terms in $x_1(\tau_1, \tau_2)$ by requiring

$$2\dot{B} + B = 0, \qquad 2\dot{A} + A = 0.$$

Along with the initial conditions in (30), we find that $B \equiv 0$ and $A(\tau_2) = \exp(-\tau_2/2) = \exp(-\varepsilon t/2)$.

This gives us the zeroth order approximation

$$x(t) \approx x_0(t) = e^{-\varepsilon t/2}\sin(t).$$

## 19. Lecture XIX: Perturbation Theory II

Today we shall consider the application of perturbation theory to oscillatory solutions of the van der Pol system that we encountered in Example 7.1. We shall be interested in applying to it the Poincaré-Lindstedt method in a small parameter regime. We shall also examine perturbation of systems dependent on a small parameter where the behaviour changes dramatically when $\varepsilon = 0$ is reached — systems for which there is some the limit behaviour is not well approximated by behaviours of system with small but non-zero $\varepsilon$. Such systems are not uncommon. Mildly viscous and entirely inviscous fluids behave very differently, for example, because a parabolic equation (which governs viscous flows),

$$\partial_t u - \varepsilon \Delta u + u \partial_x u = f(t, x, u)$$

becomes a hyperbolic equation in the $\varepsilon \to 0$ limit.

19.1. **Lienard's Theorem.** Before we continue, however, it is necessary to show that the van der Pol system has periodic/oscillatory solutions. Even though we have discussed planar systems at length, and even considered the van der Pol system in the neighbourhoods of its hyperbolic fixed points, we have yet to acquire the tools that shall allow us to show that the van der Pol system has periodic solutions. We shall therefore be taking a small detour into the theory of Lienard systems.

A Lienard equation is an equation of the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = 0.$$

Using the transformation

$$F(x) = \int_0^x f(r)\,\mathrm{d}r, \qquad x = x, \quad y = \frac{\mathrm{d}x}{\mathrm{d}t} + F(x),$$

we arrive at a system of the following form,

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x). \end{aligned}$$



Such systems are known as Lienard systems.

Suppose that $F$ and $g$ satisfy the following conditions:

(i) $g, F \in C^1(\mathbb{R})$,
(ii) $g$ and $F$ are odd,
(iii) $xg(x) > 0$ for $x \neq 0$
(iv) $F'(0) < 0$,
(v) $F$ has a single zero at $a > 0$, and
(vi) $F$ increases monotonically to infinity for $x \geq a$.

**Theorem 19.1** (Lienard's Theorem)**.** *A Lienard system satisfying the conditions (i) - (vi) has a unique limit cycle, and that limit cycle is stable.*

As in many of the "energy"- or "harmonic"-type methods (Lyapunov function, Hamiltonian function, function from Dulac's theorem) we have used to analyse the workings of the centre manifold, this theorem can be proven with a suitably chosen energy. Our candidate is

$$u(x, y) = \frac{1}{2}y^2 + G(x), \qquad \int_0^x g(w)\,\mathrm{d}w.$$

which is very recognizably the total energy, with a "kinetic" part, $y^2/2$, and a "potential" part $G$, with corresponding "force", $g$.

*Proof.* Since $F$ and $g$ are odd, it holds that $F(0) = g(0) = 0$. As well, $F(-a) = F(a) = 0$. Condition (ii) implies that $g(x) > 0$ for $x > 0$ and $g(x) < 0$ for $x < 0$.

Therefore any trajectory intersecting the upper or lower $y$-axis must intersect it perpendicularly ($\dot{y} = 0$). Any trajectory must also intesect the curve $y = F(x)$ vertically as $\dot{x} = 0$ at the intersection.

Notice that in the right half-plane, $\dot{y} < 0$, and above the curve $y = F(x)$, $\dot{x} > 0$, whilst below this curve $\dot{x} < 0$. Also notice that the transformation $(x, y) \mapsto (-x, -y)$ leaves the system unchanged.

These three things imply:

(i) that a trajectory starting at $P_0 = (0, y_0)$ on the upper $y$-axis will intersect $y = F(x)$ at $P_1 = (x_1, y_1)$ first, and then $P_2 = (0, y_2)$ on the lower $y$-axis,
(ii) this arc of the trajectory in the right-half plane is bounded by the rectangle $[0, x_1] \times [y_2, y_0]$,
(iii) and that this trajectory will continue into a closed curve if and only if $y_2 = -y_0$.

For the energy function $u(x, y)$, the closed curve condition is equivalent to $u(0, y_0) = u(0, y_2)$. Since the system is autonomous, for each point $(x_1, F(x_1))$, we can find a unique pair $(y_0(x_1), y_2(x_1))$ such that the trajectory passing through $(x_1, F(x_1))$ also passes through $(0, y_0)$ and $(0, y_2)$ with no other intersection with the $y$-axis in between. Define now

$$\varphi(x_1) = u(0, y_2(x_1)) - u(0, y_0(x_1)) = \int_A \mathrm{d}u,$$

where $A$ is the arc of the trajectory described, between $(0, y_0)$ and $(0, y_2)$.

Notice that

$$\mathrm{d}u(x, y) = (y\dot{y} + g(x)\dot{x})\mathrm{d}t = -F(x)g(x)\ \mathrm{d}t.$$

If $x_1 \leq a$, then the entire arc is bounded between the $y$-axis and $y = a$. Therefore, $F < 0$ on that arc, and $g > 0$, so that $\mathrm{d}u > 0$, and $\phi(x_1) > 0$, so a trajectory passing through such a point $(x_1, F(x_1))$ cannot be not closed. Furthermore $\varphi(a) > 0$.

We shall show that $\varphi$ decreases monotonically to $-\infty$ from $\varphi(a) > 0$ on $[a, -\infty)$, which would imply the existence of a value $\alpha$ for which $\varphi(\alpha) = 0$.

We can decompose $A$ into three arcs, $A_1$ from $(0, y_0)$ to $(a, v_1)$, $A_2$ from $(a, v_1)$ to $(a, v_2)$, and finally, $A_3$ from $(a, v_2)$ to $(0, y_2)$. Along the first and last arcs,

$$\int_{A_1} \mathrm{d}u = \int_0^a -g(x)F(x)y - F(x)\ \mathrm{d}x, \qquad \int_{A_2} \mathrm{d}u = \int_a^0 -g(x)F(x)y - F(x)\ \mathrm{d}x = \int_0^a \frac{-g(x)F(x)}{-y + F(x)}\ \mathrm{d}x.$$

Both of these decrease monotonically as $x_1$ increases beyond $\alpha$ because $y$ increases in the upper arc for each $x$, and decreases in the lower arc for wach $x$. Along the arc $A_2$,

$$\int_{A_2} \mathrm{d}u = \int_{v_1}^{v_2} -g(x)F(x)\frac{\mathrm{d}t}{\mathrm{d}y}\ \mathrm{d}y = \int_{v_2}^{v_1} -F(x)\ \mathrm{d}y.$$

As $x_1$ increases, the arc moves to the right, so $F(x)$ increases monotonically to infinity, as well, $v_1 - v_2$ also increases to infinity. Therefore the integral decreases monotonically to infinity, and there are no cancellations from the other arcs.

The sign of $\varphi(x_1)$ on either side of $\varphi^{-1}(0)$ implies stability.

□

Recall from Example 7.1 that the van der Pol system is

$$\dot{x} = y$$
$$\dot{y} = -\beta(x^2 - 1)y - x.$$

We can put this into Lienard form by setting $z = y + F(x)$, where $f(x) = \beta(x^2 - 1)$. Now that we know there is a limit cycle, we shall try to approximate it and analyse solutions about it.

19.2. **Revisiting the Poincaré-Lindstedt Method.** We shall write the van der Pol system as a scalar equation

$$\ddot{x}(t) + \beta(x^2 - 1)\dot{x}(t) + x(t) = 0, \qquad x(0) = a,\ \dot{x}(0) = 0.$$

Suppose $\beta > 0$ is small, and take $\tau = \omega(\beta)t$, where $\omega(\beta) = 1 + \beta\omega_1 + \beta^2\omega_2 + \cdots$. The van der Pol equation then becomes

$$\omega^2(\beta)\ddot{x}(\tau) + \beta\omega(\beta)(x^2 - 1)\dot{x}(\tau) + x(\tau) = 0.$$

Using the ansatz

$$x(\tau) = x_0(\tau) + \beta x_1(\tau) + \beta^2 x_2(\tau) + \cdots,$$

we have the initial conditions

$$x_0(0) = a, \quad 0 = \dot{x}_0(0) = x_1(0) = \dot{x}_1(0) = \cdots.$$

We can collect like powers of $\beta$ and arrive at the equations:

$$\begin{aligned}
&\ddot{x}_0(\tau) + x_0(\tau) = \\
&\ddot{x}_1(\tau) + x_1(\tau) = -2\omega_1\ddot{x}_1(\tau) - (x_0^2(\tau) - 1)\dot{x}_1(\tau) \\
&\ddot{x}_2(\tau) + x_2(\tau) = -\big((\omega_1^2 + 2\omega_2)\ddot{x}_0(\tau) + 2\omega_1\ddot{x}_1(\tau)\big) - \big(x_0^2(\tau) - 1\big)\big(\dot{x}_1(\tau) + \omega_1\dot{x}_0(\tau)\big) \\
&\qquad\qquad - 2x_0(\tau)x_1(\tau)\dot{x}_0(\tau) \\
&\qquad \ldots.
\end{aligned}$$

As before have

$$x_0(\tau) = a\cos(\tau).$$

Substituting this solution into the second equation,

$$\ddot{x}_1(\tau) + x_1(\tau) = 2a\omega_1\cos(\tau) - a(1 - a^2/4)\sin(\tau) + (a^3/4)\sin(3\tau).$$

The degrees of freedom provided by the free choices of $\omega_i$ allow us to eliminate secular terms.

Looking more closely at the Poincaré-Lindstedt method, we shall see that if we tried to elimate resonant forces/secular terms from a non-periodic solution, we shall be unable so to do. For one of the resonant terms can be elimated by choosing $\omega_1 = 0$. But the other requires the choice $a = \pm 2$, which are the initial conditions that lands us on the limit cycle. That gives us $x_1(\tau) = \sin^3(\tau)$.

Let us consider one higher order in the expansion $\omega(\beta)$. Substituting $x_1(\tau)$ into the third equation, we arrive at

$$\begin{aligned}
\ddot{x}_2(\tau) + x_2(\tau) &= 2a\omega_2\cos(\tau) - (a^2\cos(\tau) - 1)3\sin^2(\tau)\cos(\tau) \\
&\quad - 2a^2\cos(\tau)\sin^4(\tau) \\
&= (4\omega_2 - 11)\cos(\tau) - 31\cos^3(\tau) + 20\cos^5(\tau) \\
&= \left(4\omega_2 - \frac{1}{4}\right)\cos(\tau) - \frac{3}{2}\cos(3\tau) + \frac{5}{4}\cos(5\tau).
\end{aligned}$$

Therefore we should take $\omega_2 = -1/16$, and have, up to order $\beta$,

$$x(t) = 2\cos(\omega t) + \beta\sin^3(\omega t) + O(\beta^2), \qquad \omega = 1 - \beta^2/16 + \cdots.$$

As expected, with $\beta$ small, the limit cycle is close to a circle of radius $a = 2$.

19.3. **The WKB method.** What if $\beta$ were large? Scaling time by $t/\beta \mapsto t$ (the slow time scale), we can consider the scaled equation

$$\frac{1}{\beta^2}\ddot{x} + (x^2 - 1)\dot{x} + x = 0.$$

The limit $\beta \to \infty$ is singular because it doesn't have oscillatory behaviour and we do not expect its solutions to approximate limit cycles of systems with large but finite $\beta$ except for very short times. And indeed, for a short enough time, there is a solution to the truncated, integrable equation

$$(x^2 - 1)\dot{x} + x = 0,$$

which breaks down in finite time.

There is a fairly popular perturbation method that handles vanishing highest-order terms known as the WKB method, or the Carlini-Liouville-Green method. However, it is generally restricted to equations of the form

$$\varepsilon^2\ddot{x}(t) + V(t)x(t) = 0.$$

Let us step back from the van der Pol system and consider simpler systems of the above form, where $V$ has isolated roots.

Suppose $V$ were constant. Then we expect

$$x(t) = C_1 e^{-\sqrt{V}t/\varepsilon} + C_2 e^{\sqrt{V}t/\varepsilon}.$$

Comparing this to Example 18.3, it is evident that a vanishing highest order term gives dynamics that are very different from a vanishing lower-order term. The particular feature of the WKB method is the assumption of decay in the fast time variable $\tau_1 = t/\varepsilon$, so that ansatz are taken of the form:

$$x(t) = e^{\vartheta(t)/\varepsilon^\alpha}\big(x_0(t) + \varepsilon^\alpha x_1(t) + \cdots\big).$$

Taking derivatives of this ansatz, we find that

$$\begin{aligned}\dot{x} &= e^{\vartheta/\varepsilon^\alpha}\big(\varepsilon^{-\alpha}\dot{\vartheta}x_0 + \dot{x}_0 + \dot{\vartheta}x_1 + \varepsilon^\alpha\dot{x}_1 + \cdots\big)\\ \ddot{x} &= e^{\vartheta/\varepsilon^\alpha}\big(\varepsilon^{-2\alpha}\dot{\vartheta}^2x_0 + \varepsilon^{-\alpha}(\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1) + \cdots\big),\end{aligned}$$

etc., and upon substitution into the equation we have

$$\varepsilon^2\left[\varepsilon^{-2\alpha}\dot{\vartheta}^2x_0 + \varepsilon^{-\alpha}\big(\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1\big) + \cdots\right] + V(t)(x_0 + \varepsilon^\alpha x_1 + \cdots) = 0.$$

The linearity of the equation allowed us to multiply through with $\exp(-\vartheta(t)\varepsilon^{-\alpha})$.

Setting $\alpha = 1$ allows us to collect terms of like powers. The zeroth order equation is

$$\dot{\vartheta}^2 = V.$$

This can be integrated directly.

The first order equation is

$$\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 + \dot{\vartheta}^2x_1 - V(t)x_1 = 0.$$

Using the first equation, we find that we need only solve

$$\ddot{\vartheta}x_0 + 2\dot{\vartheta}\dot{x}_0 = 0,$$

which has the solution

$$x_0(t) = \frac{c}{\sqrt{\dot{\vartheta}}}.$$

These give a first order approximation of

$$x(t) = V^{-1/4}(t)\left[C_0\exp\left(\frac{-1}{\varepsilon}\int^x\sqrt{V}(s)\,\mathrm{d}s\right) + C_1\exp\left(\frac{1}{\varepsilon}\int^x\sqrt{V}(s)\,\mathrm{d}s\right)\right].$$

## 20. Lecture XX: Index Theory I

20.1. **Notion and Properties of the Index.** Index theory in dynamical systems is another manifestation of a phenomenon you may have encountered before in, say, residue theory/winding number/monodromy in complex analysis, or Euler-Poincaré indices of compact 2-D (or higher dimensional) manifolds or graphs, or degree theory of differentiable maps, or indeed, index theory of linear maps in Fourier analysis (Toeplitz operators).

The basic idea is that it is often possible to assign an integer invariant to maps, motivated by algebraic, geometric, or topological considerations, that proves remarkably useful, for example, in distinguishing one type of (nonlinear) dynamics from another.

Let us begin with some definitions. The INDEX of a Jordan curve $C \subseteq \mathbb{R}^2$ on a vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (P(x, y), Q(x, y))^\top$, with no critical point on $C$ is defined as

$$I_f(C) := \frac{1}{2\pi} \oint_C \mathrm{d}\Big( \arctan(\frac{\mathrm{d}y}{\mathrm{d}x}) \Big) = \frac{1}{2\pi} \oint_C \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2},$$

the second equality following from L'Hospital's rule. This is the averaged curl around the curve $C$, the amount of circulation picked up along $C$, reminiscent of Green's theorem as well as of Thm.13.1. From arctangent definition, and from the fact that $f$ is $C^1$, with none of its critical points on the curve $C$, it can be seen that $I_f(C)$ must needs be an integer (because $f$ varies smoothly along $C$).

We say that a(n oriented) Jordan curve $C$ is the sum of two (oriented) Jordan curves $C_1$ and $C_2$, written as $C = C_1 + C_2$, if $C_1$ and $C_2$ (are compatibly oriented) and share a line segment. In a theorem reminscent of the Cauchy-Goursat integral theorem,

**Theorem 20.1.** 1. *Let a Jordan curve $C$ be the sum of Jordan curves $C_1$ and $C_2$. Then with respect to any $C^1$ vector field $f$ with no critical point on $C_1$ or $C_2$,*

$$I_f(C) = I_f(C_1) + I_f(C_2).$$

2. *If neither $C$ nor its interior contain a critical point of $f$, then $I_f(C) = 0$.*



*Proof. 1.*

This is a direct consequence of the linearity of integrals.

*2.*

Suppose $P^2 + Q^2 > \sigma > 0$ is bounded away from nought on an open set $E$ by the lack of critical points of the system/vector field $f$ on $E$. Then for any $\varepsilon > 0$, a piecewise-$C^1$ Jordan curve $C$ entirely within $E$ of length $\delta$ may be chosen so that the quantity

$$I_f(C) = \left| \frac{1}{2\pi} \oint_C \frac{P\mathrm{d}Q - Q\mathrm{d}P}{P^2 + Q^2} \right| < \frac{\delta}{2\pi\sigma} \sup_{\mathbf{x}\in E} \left| P\frac{\mathrm{d}Q}{\mathrm{d}r} - Q\frac{\mathrm{d}P}{\mathrm{d}r} \right|, \tag{31}$$

where $\mathrm{d}/\mathrm{d}r$ is the directional derivative along $C$ in the counter-clockwise direction, is bounded by $\varepsilon$ by taking $\delta$ to be sufficiently small. This is because $P, Q \in (C^1(\mathbb{R}^2))^2$.

Since the integral $I_f(C)$ is an integer, with $\varepsilon < 1$, we can conclude that $I_f(C) = 0$.

Suppose now that $C \subseteq E$ is a bounded Jordan curve of any finite length. Denote its interior by $U$. We can divide $C \cup U$ by a finite grid of side length $\delta/4$ and apply the estimate (31) to every square lying entirely within $U$. Part 1. of this theorem then suggests that for the curve $\tilde{C}$, the sum of the perimeter of these squares, $I_f(\tilde{C}) = 0$.

Each of the remaining squares contain a portion $C$. Since $C$ is of finite length we can subdivide these squares into smaller and smaller squares only finitely many times until either a square is inside $U$ entirely or a square $S$ contains a portion of $C$ shorter than $\delta$. Then the reduced square $S\backslash(C \cup U)^c$ has a boundary that is piecewise $C^1$ and of length $2\delta$. By choosing $\varepsilon < 1/2$, we can still use (31) to conlude that the remaining fintely many squares and reduced squares have index zero.

□

**Corollary 20.2.** *Let $C_1$ and $C_2$ be Jordan curves for which $C_2$ lies in the interior of $C_1$ and there exists only one critical point of the $C^1$-vector field $f$ on the union of $C_1$ and its interior, which also lies in $C_2$. Then $I_f(C_1) = I_f(C_2)$.*

*Proof.* Let $p_1, p_2$ be distinct points on $C_1$ and $q_1$ and $q_2$ be distinct points on $C_2$. Let the counter clockwise arcs on $C_1$ and $C_2$ be defined:

$$A_1 = p_1\widehat{p}_2, \quad A_2 = p_2\widehat{p}_1, \quad B_1 = q_1\widehat{q}_2, \quad B_2 = q_2\widehat{q}_1.$$

Let $M_1 = \overline{p_1q_1}$ and $M_2 = \overline{p_2q_2}$ be two line segments that do not intersect one another or the interior of $C_2$.

The corollary statement is immediate as we can consider the sum of three Jordan curves whose enclosed areas do not intersect:

$$\Gamma_1 = A_1M_2\bar{B}_1\bar{M}_1, \qquad \Gamma_2 = A_2M_1\bar{B}_2\bar{M}_2,$$

and $C_2$, defined counter clockwise. Here we use the over bar to denote the same arc in the reverse sense. We see that $\Gamma_1 + C_2 + \Gamma_2 = C_1$.

The theorem then tells us that

$$I_f(\Gamma_1) + I_f(C_2) + I_f(\Gamma_2) = I_f(C_1),$$

and since $\Gamma_1$ does not enclose or include a critical point of $f$, $I_f(\Gamma_1) = 0$. Likewise, $I_f(\Gamma_2) = 0$. □

This implies that any Jordan curve enclosing a unique critical point $\mathbf{x}_0$ of a vector field $f \in (C^1(\mathbb{R}^2))^2$ has the same index. Therefore it makes sense to define

$$I_f(\mathbf{x}_0) := I_f(C),$$

where $C$ is a $C^1$-Jordan curve enclosing the unique critical point $\mathbf{x}_0$.

Using a similar construction to the above, we also have

**Corollary 20.3.** *Let $C$ be a Jordan curve enclosing $n$ isolated critical points $\{\mathbf{x}_i\}_{i=1}^n$ of $f \in (C^1(\mathbb{R}^2))^2$ and only these critical points. Then*

$$I_f(C) = \sum_{i=1}^{n} I_f(\mathbf{x}_i).$$



One simply considers $C$ in place of $C_1$ above and a collection $\{J_i\}_{i=1}^n$ of $n$ mutually disjoint Jordan curves, each enclosed in $C$ with $J_i$ enclosing $\mathbf{x}_i$, so that $I_f(C_i) = I_f(\mathbf{x}_i)$, in place of $C_2$ above.

20.2. **Applications of the Index.** Till now we have yet to apply the index to describe actual systems.

Since the previous results suggest that there is some robustness to the value of the index, it is no surprise that

**Theorem 20.4.** *If a $C^1$-first order planar system is governed by*

$$f(\mathbf{x}) = \mathrm{D}f|_{\mathbf{x}_0} + g(\mathbf{x} - \mathbf{x}_0),$$

*where $\mathbf{x}_0$ is an isolated critical point, and $|g(\mathbf{x})| = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to 0$, then*

$$I_f(\mathbf{x}_0) = I_v(\mathbf{x}_0),$$

*where $v$ is the vector field given by the linearisation, $v(\mathbf{y}) = \mathrm{D}f|_{\mathbf{x}_0}\mathbf{y}$ for $\mathbf{y} \in \mathbb{R}^2$.*

Next we look at some results on calculation of indices and what the values of indices imply about the (local) vector fields they are defined over.

**Theorem 20.5.** *Let $f \in (C^1(E))^2$ be a vector field on an open subset $E \subseteq \mathbb{R}^2$. Let $\Gamma$ be a periodic orbit of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ lying entirely in $E$. Then $I_f(E) = 1$.*

*Proof.* This is a very simple calculation in polar coordinates. Let $\Gamma$ be parameterized by $(r, \vartheta)$, and have period $T$:

$$\begin{aligned} I_f(\Gamma) &= \frac{1}{2\pi} \oint_\Gamma \mathrm{d}\Big( \arctan(\frac{\mathrm{d}y}{\mathrm{d}x}) \Big) \\ &= \frac{1}{2\pi} \int_0^T \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r\cos(\theta)\mathrm{d}\big(r\sin(\theta)\big) - r\sin(\theta)\mathrm{d}\big(r\cos(\theta)\big)}{r^2} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta)\mathrm{d}\sin(\theta) + \frac{1}{r}\cos(\theta)\sin(\theta)\mathrm{d}r - \sin(\theta)\mathrm{d}\cos(\theta) - \frac{1}{r}\sin(\theta)\mathrm{d}\cos(\theta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\cos^2(\theta) + \sin^2(\theta))\mathrm{d}\theta \\ &= 1. \end{aligned}$$

□

The theorem above, along with part 2 of Thm.20.1 clearly implies

**Corollary 20.6.** *Let $f \in (C^1(E))^2$ be a vector field on an open subset $E \subseteq \mathbb{R}^2$. Let $\Gamma$ be a periodic orbit of the system $\dot{\mathbf{x}} = f(\mathbf{x})$ lying entirely in $E$. Then there is at least one critical point of $f$ in the interior of $\Gamma$.*

Recall that in Lecture 13, we discussed planar systems for which $\mathrm{D}f|_{\mathbf{x}_0}$ at an isolated critical point $\mathbf{x}_0$ was degenerate. It either had one zero eigenvalue, two zero eigenvalues but a geometric multiplicity of one, or two zero eigenvalues with full geometric multiplicity. Whilst we characterized behaviours in the first two cases, we left the final case open. In the final case, it is possible for there to be many (compatibly oriented) hyperbolic, elliptic, and parabolic sectors at an isolated critical point. We shall address some aspects of the final case now.

**Theorem 20.7** (Bendixson's Index Formula)**.** *Let $e$ be the number of elliptic sectors and $h$ be the number of hyperbolic sectors at an isolated critical point $\mathbf{x}_0$ of a dynamical system governed by the analytic vector field $f$. Then*

$$I_f(\mathbf{x}_0) = 1 + \frac{1}{2}(e - h).$$

This theorem is a consequence of the following lemma, which is also helpful in the effective calculation of the index:

**Lemma 20.8.** *Let $\mathbf{x}_0$ be an isolated critical point of the planar analytic system*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$

*Let $C$ be a Jordan curve enclosing $\mathbf{x}_0$ and no other critical points. Let $A$ be the number of times $Q(x, y)/P(x, y)$ jumps from $-\infty$ to $\infty$ and let $D$ be the number of times $Q(x, y)/P(x, y)$ jumps from $\infty$ to $-\infty$ as $(x, y)$ varies one cycle along $C$ in the counter-clockwise sense. Then*

$$I_{(P,Q)^\top}(\mathbf{x}_0) = \frac{1}{2}(D - A).$$



*Proof.* The index is defined by the formula

$$I_{(P,Q)^\top}(\mathbf{x}_0) := \oint_C \mathrm{d}\Big(\arctan(\frac{\mathrm{d}y}{\mathrm{d}x})\Big).$$

By L'Hospital's rule, $\mathrm{d}y/\mathrm{d}x = Q/P$, so setting

$$\Theta(x, y) = \arctan(\frac{Q(x, y)}{P(x, y)}), \tag{32}$$

we have

$$I_{(P,Q)^\top}(\mathbf{x}_0) := \frac{1}{2\pi} \oint_C \mathrm{d}\Theta(x, y).$$

Next notice that $|Q(x, y)/P(x, y)|$ is infinite on $C$ exactly when $P(x, y) = 0$ because $P$ and $Q$ are never zero cotemporaneously on $C$.

Now every time $Q(x, y)/P(x, y)$ jumps from $-\infty$ to $\infty$, it crosses an angle $(2k + 1)\pi/2$ clockwise (decreases across such an angle). Similarly, every time $Q(x, y)/P(x, y)$ changes from $\infty$ to $-\infty$, it crosses an angle $(2k + 1)\pi/2$ counter clockwise (increases across such an angle).

As the integral is phase-independent, we can add any fixed small shift $\vartheta$ to $\Theta$, and see that every increase in $2\pi$ of $\Theta$ involves two jumps from $\infty$ to $-\infty$, and every decrease in $2\pi$ of $\Theta$ involves two jumps from $-\infty$ to $\infty$.

□

What Bendixson's index theorem does suggest is that it is possible to calculate an index by inspecting the phase portrait, and this lemma foregoing allows us to do so algebraically from the equations.

*Remark* 20.1. Assuming the hypotheses of the lemma, let $M$ and $N$ respectively be the number of times that $Q(x, y)/P(x, y)$ changes sign along $C$ at a zero of $Q$ from negative to positive, and from positive to negative. Then

$$I_{(P,Q)^\top}(\mathbf{x}_0) = \frac{1}{2}(M - N).$$

■

*Remark* 20.2. Recall that the winding number of a curve $\gamma$ on $\mathbb{C}$ about a point is defined as

$$W(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi} \int_\gamma \mathrm{d} \arg(z - z_0).$$

From the calculations in the proof above (specifically (32)), it is also apparent what is meant that if the vector field $f$ were "straightened-out", so that corresponding contortions were introduced to a simple closed curve $\gamma$ on the same plane on which lies $f$, and thereafter the plane were identified canonically with $\mathbf{C}^2$, then $I_f(\gamma)$ is the winding number of the contorted curve. The plane straightened out by $f$ is simply the plane where $f(z)$ replaces $z$.

■

## 21. Lecture XXI: Index Theory II

In this lecture we shall calculate some examples where we apply Bendixson's Index Theorem, or the attendent lemma. Next we shall look at another way to technique by which to analyse planar systems globally, much like we used the polar coordinates transformation to analyse them locally.

### 21.1. **Examples.**

**Example 21.1.** Consider the system

$$\dot{x} = P(x, y) = x,$$
$$\dot{y} = Q(x, y) = -y.$$

This is the archetypal saddle. There is a critical point at the origin and it is a saddle.

Using Lemma 20.8, or the remark thereafter, let us take a small simple closed curve about the origin. For concreteness, we can take the circle of radius $\delta > 0$. As we traverse this circle in the counter-clockwise sense, we encounter two places where $Q(x, y) = 0$, namely the intersections with the positive and negative parts of the $x$-axis. In order to avoid having any of these intersections at the end-points of our traversal on the circle, and introducing some ambiguity as to the number of intesections encountered, we can start at, say, $\vartheta = \pi/4$.

At the first encounter with $Q(x, y) = 0$, we are at $(-\delta, 0)$. The sign of $Q(x, y)/P(x, y)$ changes from positive to negative.

At the second encounter with $Q(x, y) = 0$, we are at $(\delta, 0)$. The sign of $Q(x, y)/P(x, y)$ changes from positive to negative.

We conclude that

$$I_f(\mathbf{0}) = \frac{1}{2}(0 - 2) = -1.$$



**Example 21.2.** Consider the nonlinear system

$$\dot{x} = P(x, y) = x^2 + xy$$
$$\dot{y} = Q(x, y) = \frac{1}{2}y^2 + xy.$$

This system can be found on pp 151 - 152 of *Perko*. It has a critical point at the origin with two elliptic sectors and two parabolic sectors.

We see that $Q(x, y) = 0$ along $y = 2x$ and $y = 0$. A circle enclosing the origin traverses this line twice. Going in a counter-clockwise sense, starting at $\vartheta = \pi/4$, at the first intersection we find $Q(x, y)/P(x, y)$ changes sign from negative to positive. This happens again at each of the subsequent intersections. Therefore,

$$I_f(\mathbf{0}) = \frac{1}{2}(4 - 0) = 2,$$

also verifying Bendixson's index theorem.

### 21.2. **The Poincaré Index Theorem.**

The Poincaré index theorem equates the index of a system on a two-dimensional surface and the Euler-Poincaré characteristic of that surface. A two dimensional surface is a two dimensional, compact, $C^2$-manifold, defined in a manner described in Lecture 8. Alternatively, for us, it is the surface of a doughnut with a finite number of holes, although we make the caveat that this alternative excludes a number of interesting two dimensional surfaces not smoothly embeddable in $\mathbb{R}^3$, in particular, to non-orientable surfaces, to which the theorem still in fact applies. It goes beyond the scope of this module to discuss what precisely orientability means. But we point out that smooth orientable 2-dimensional surfaces are all smoothly embeddable in $\mathbb{R}^3$.

On every neighbourhood of such a surface $M$, we can locally parameterize the surface by coordinates $(x, y)$, as a two dimensional manifold "looks locally like $\mathbb{R}^2$". And it makes sense to discuss a two-dimensional dynamical system on this surface, just as we had done for a system on a plane.

The vector field $f(p)$ will be in a separate vector space, the tangent space $T_pM$. Suppose now that this vector field has finitely many critical points $\{\mathbf{p}_i\}_{i=1}^m$.

Using the local homeomorphism to $\mathbb{R}^2$, around each point $\mathbf{p}_i$, there is a small neighbourhood $U_i$ of $\mathbf{p}_i$ and a smooth homeomorphism $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^2$ between $U_i$ and an open neighbourhood in $\mathbb{R}^2$. We can define the index $I_f(\mathbf{p}_i)$ around the critical point $\mathbf{p}_i$ on the surface $M$ as the index $I_{f_j}(\varphi_i(\mathbf{p}_i))$ of the vector field $f_i := \varphi_{i*}f$ (the pushfoward is defined in Lecture 8, which is $f$, written out in local coordinates $\varphi_i$ — for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$, $(\varphi_{i*}f)g = f^j\partial_j(g \circ \varphi_i)$ in coordinates).

A heuristic way to think about this is that $U_i$ is basically a flat patch, and the index at $\mathbf{p}_i$ can be found by drawing a small enough simple closed curve around $\mathbf{p}_i$ so that the curve fits inside $U_i$. We need coordinates $\varphi_i$ essentially so that we can speak clearly about the line integral defining the index.

We then define the INDEX OF THE SURFACE $M$ relative to the vector field $f$ as

$$I_f(M) := \sum_i I_f(\mathbf{p}_i).$$

The startling fact is that $I_f(M)$ is in fact vector field independent. This number only depends on $M$, and is equal to the EULER-POINCARÉ CHARACTERISTIC of $M$, $\chi(M)$. This number $\chi(M)$ is calculated in the following way:

First we invoke the fact that any two dimensional surface can be tessellated by finitely many curvilinear triangles. These curvilinear triangles may be found by first tessellating a local patch $V_i = \varphi_i(U_i) \subseteq \mathbb{R}^2$ by actual triangles in $\mathbb{R}^2$, which defines the triangle on $U_i \subseteq M$. There are two things of which we want to be sure. The first is that these tessellations match up properly. The second is more notational — I used $U_i$ and $V_i$ because we had been talking about them, but in fact, we can find patches to cover all of $M$, and not just near critical points of some arbitrary vector field $f$. This tessellation is known as a TRIANGULATION. Given a triangulation $\Delta$ of a surface, we can calculate an integer

$$\chi_\Delta(M) = F - E + V,$$

where $F$ is the number of faces, $E$ the number of edges, and $V$ the number of vertices.

A triangulation is not unique. However, by varying any given triangulation, say, by removing edges and vertices, it can be shown that any triangulation returns the same number $\chi_\Delta(M)$, so in fact $\chi_\Delta(M)$ is independent of the triangulation $\Delta$, and we can call it $\chi(M)$. In fact, by removing edges, it can be shown that this number does not depend on the tessellation being one of curvilinear triangles at all. $\chi(M)$ only depends on topological properties of $M$. We call such a number a TOPOLOGICAL INVARIANT.

Having defined both $I_f(M)$ and $\chi(M)$, we can state the following theorem:

**Theorem 21.1** (Poincaré Index Theorem)**.** *Let $M$ be a smooth two-dimensional surface, and $f$ a $C^1$-vector field defined on it, with at most finitely many critical points. Then*

$$I_f(M) = \chi(M).$$

Refer to *Perko*, pp.307 – 310 for a sketch of a proof. The idea for orientable surfaces is to show it for a sphere by reasoning about curves on spheres and the resulting index, and then proceed to add "handles" to the sphere, where each "handle" is itself homeomorphic to a sphere.

21.3. **Compactifying the plane.** This brings us to a notion that is important in the theory of planar systems but which we shall not have time to explore in detail.

It is possible to compactify a plane by adding a point at infinity. This notion on the complex plane of adding a "point at infinity" turns the complex plane into closed surface known as the Riemann sphere. As you may also be familiar via the theory of fractional linear transformation (i.e., Möbius transformations), it is possible to project the plane onto a sphere $\{(x, y, z) : x^2 + y^2 + (z-1)^2 = 1\}$, sitting atop the plane, tangent to it at the origin, via the standard stereographic projection. This

projects the point at infinity to the "north pole", and gives us a way to discuss criticality at infinity. Where the $xy$-plane is the phase plane of a planar system, we call this sphere the BENDIXSON SPHERE.

Another projection that is often used generates the POINCARÉ SPHERE. This projection projects $\mathbb{R}^2$ again onto the sphere $\{(x,y,z) : x^2+y^2+(z-1)^2 = 1\}$, but this time, we take a line from the centre of the enclosed ball, $(0,0,1)$. We associate the intersection of this line with the $xy$-plane with the point of intersection of this line with the sphere that is not the north pole. For simplicity of calculations, it helps to consider the change-of-coordinates $(x,y,z) \mapsto (X,Y,Z)$ on the sphere, where $X = x$, $Y = y$, and $Z = 1-z$. In the new coordinates, the sphere is $\{(X,Y,Z) : X^2+Y^2+Z^2 = 1\}$. Let us use small letters for the phase plane and capital letters for the coordinates on the sphere. The map $(x,y) \mapsto (X,Y,Z)$ from the intersection with the plane to the associated point on the sphere can be written out explicutly:

By looking at similar triangles, it is clear that

$$\frac{x}{1} = \frac{X}{Z}, \qquad \frac{y}{1} = \frac{Y}{Z},$$

and using $X^2+Y^2+Z^2 = 1$, we find that

$$(xZ)^2 + (yZ)^2 + Z^2 = 1,$$

so we can write the projection as

$$X = \frac{x}{\sqrt{1+x^2+y^2}}, \qquad Y = \frac{y}{\sqrt{1+x^2+y^2}}, \quad Z = \frac{1}{\sqrt{1+x^2+y^2}}.$$

The idea of the Poncaré sphere is that in the Bendixson projection, the behaviour at infinity is projected to a point, but on the Poincaré sphere, it is spread out over the equator, and so may become less complicated.

Either way, we see that starting with a planar system, we can find a homeomorphic system on a compact two-dimensional surface. Therefore we have an index. Now if $f$ on the plane had finitely many critical points, we have seen that the sum of their indices can be any integer. But the Euler-Poincaré characteristic of a sphere (as of a cube, or a dodecahedron) is $\chi(S^2) = 2$.

To reconcile the discrepancy we have to notice that it is possible to have a critical point at infinity. We should like to be able independently to calculate the index at infinity to be able to show that adding it to the indices at the remaining critical points, we in fact arrive at the Euler-Poincaré characteristic of a sphere, 2.

To this end it is actually simpler to look at not the Poincaré or Bendixson spheres, but simply an inversion of the plane by the unit circle (flipping the plane inside-out about the unit circle).

Let us define the inverted coordinates of the $xy$-plane by

$$X = \frac{x}{x^2+y^2}, \qquad Y = \frac{-y}{x^2+y^2}.$$

The point of an inversion is that $(X^2+Y^2)\cdot(x^2+y^2) = 1$ with a reflection. So in polar coordinates $(r,\vartheta)$, an inversion is given by

$$R = \frac{1}{r}, \qquad \Theta = -\vartheta.$$

(Perhaps see Coxeter's *Introduction to Geometry* if you have not heard of inversions in Euclidean geometry classes.)

This is familar as the inversion from complex analysis if we set $z = x+iy$, $Z = X+iY$, then an inversion is

$$Z = \frac{1}{z},$$

which accounts for the prima facie superfluous reflection.

Writing $f = (P, Q)^\top$, set $g = P + iQ$, where we abuse notation so that $P(x, y) = P(z)$, and mutatis mutandis for $Q$, $f$ and $g$. If we invert $g$ by

$$G = \Big(\frac{P}{P^2 + Q^2}, \frac{-Q}{P^2 + Q^2}\Big)^\top,$$

we can readily see from the linearity of the derivative that the inverted system to

$$\dot{z} = P(z) + iQ(z)$$

is

$$\dot{Z} = G_1(1/Z) + iG_2(1/Z),$$

where, of course, $G = G_1 + iG_2$. The inverted system in polar coordinates is simply

$$\dot{R} = \frac{-\dot{r}}{r^2}, \qquad \dot{\Theta} = -\dot{\vartheta}.$$

We define the INDEX AT INFINITY, denoted by $I_f(\infty)$, as the index of the origin in the inverted system.

As expected from the Poincaré index theorem, we have the following result:

**Theorem 21.2.** *Let $f$ be a $C^1$-vector field for which the planar system $\dot{\mathbf{x}} = f(\mathbf{x})$ has $m$ critical points, with indices at these critical points given by $\{I_i\}_{i=1}^m$. Then*

$$I_f(\infty) + \sum_i I_i = 2.$$

This turns on a simple calculation which itself turns on the fact that a simple closed curve integrated one way in the original system maps onto a simple closed curve oriented in the opposite sense in the inverted system. For details see *Jordan and Smith*, section 3.2.

## 22. Lecture XXII: One-Dimensional Local Bifurcations I

With this lecture we move on to the final part of the module, where we shall concern ourselves with bifurcations of systems with parameters. We have seen systems with parameters before, both fixed and vanishing. It happens that at times, the qualitative behaviour of systems change drastically and suddenly as the parameters on which the depend vary continuously. We call these phenomena BIFURCATIONS. Recall in Example 7.5, the activator-inhibitor model

$$\dot{x} = \sigma \frac{x^2}{1+y} - x$$
$$\dot{y} = \rho(x^2 - y).$$

has one critical point at the origin for any $\sigma$, but only at $\sigma > 2$ does it suddenly have two further critical points at $(r_\pm, r_\pm^2)$, where

$$r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

We say that a bifurcation occurs at $\sigma = 2$ for this system.

We shall be looking systematically at simple bifurcations in the remaining lectures in this module. First we shall consider one-parameter systems, so we shall be looking at systems of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu), \tag{33}$$

where $\mu$ varies over $\mathbb{R}$, and specifically, is *not* the temporal variable. For each fixed $\mu$, we have a $C^1$-autonomous, first order system of long familiarity. We may think of this as a system over $\mathbb{R}^{d+1}$ by augmenting it with the equation $\dot{\mu} = 0$.

22.1. **Dimension one/Codimension one bifurcations.** The reason that the title of this subsection seems somewhat of an oxymoron is that again, the nomenclature is a matter of perspective. In any case, the bifurcation is only happening along one direction. We shall introduce four types of bifurcations below.

To understand the codimension one perspective, first suppose the system (33) has a critical point at $(\mathbf{x}_0, \mu_0)$ at which $\mathrm{D}f(\mathbf{x}_0, \mu_0)$ (is an $d \times d$ matrix which) has a single zero eigenvalue. We know that hyperbolic critical points are quite stable already.

Then by the centre manifold theorem applied to the $(d+1)$ dimensional system, we know that there is a two dimensional manifold $W^c((\mathbf{x}_0, \mu_0)) \subseteq \mathbb{R}^{d+1}$ tangent to the centre subspace at $(\mathbf{x}_0, \mu_0)$. Restricting the system to the surface $W^c((\mathbf{x}_0, \mu_0))$, we can "foliate" $W^c((\mathbf{x}_0, \mu_0))$ be (one dimensional) curves indexed by a parameter $\mu$ close to $\mu_0$. (That is, we can think of $W^c((\mathbf{x}_0, \mu_0))$ as being made up entirely of curves $\gamma_\mu$ as $\mu$ varies, which do not intersect.)

The centre manifold theorem gives us:

$$\left.\frac{\partial g}{\partial y}\right|_{(\mathbf{x}_0, \mu_0)} = 0, \tag{34}$$

where $g : \mathbb{R}^{d+1} \to \mathbb{R}$ is the "centre" component of $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$, and $y$ is the "centre" variable that is not $\mu$. That is, the centre equations of the $(d+1)$-dimensional system are

$$\dot{y} = g, \qquad \dot{\mu} = 0.$$

We shall now see that by increasing the "degeneracy" of this centre manifold dynamics, we naturally arrive at different types of bifurcating behaviour, in which the system changes abruptly in different ways as $\mu$ varies continuously.

*1. Saddle-node bifurcation*

We can stem the degeneracy on the centre manifold of the $(d+1)$-dimensional system by requiring

$$\left.\frac{\partial g}{\partial \mu}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0, \qquad \left.\frac{\partial^2 g}{\partial y^2}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0. \tag{35}$$

This is a sort of transversality condition, because we know that this means whilst the curves $\gamma_\mu$ with $\mu$ close enough to $\mu_0$ all reach an optimum in the $y$ direction on $W^c((\mathbf{x}_0,\mu))$ (this is (34)), at least they do not vanish to second order in both the $y$ and the $\mu$ direction.

Integrating the equations (34) and (35) leads us directly to the equation for centre variable of the form:

$$\dot{y} = (\mu-\mu_0) - (y-y_0)^2 + O((\mu-\mu_0)(y-y_0), (\mu-\mu_0)^2, (y-y_0)^3),$$

where we have normalized over all constants that could be multiplied to $(\mu-\mu_0)$ or $(y-y_0)^2$. Here $y_0$ is the "centre" component of $\mathbf{x}_0$. The minus sign is an arbitrary convention, as we shall see.

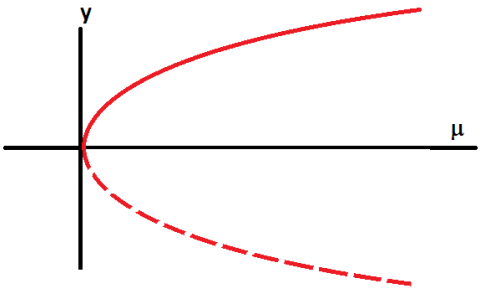
Neglecting the higher order terms, and assuming $\mathbf{x}_0 = \mathbf{0}$, $\mu_0 = 0$, there are critical points along the nullcline $y^2 = \mu$. As $\mu$ varies over $\mathbb{R}$, we see that there are no critical points for the system where $\mu < 0$. For $\mu > 0$, there are two, $y = \pm\sqrt{\mu}$. This would be reversed if the minus sign were a plus sign. At $\mu = 0$, of course, there is one single critical point, about which we had applied the centre manifold theorem to begin with.

We can ask about the stability of these two critical points where they exist. Looking again at the centre variable equation with $\mathbf{x}_0 = \mathbf{0}$ and $\mu_0 = 0$, we find that at the linearization:

$$\left.\frac{\partial}{\partial y}\right|_{\pm\sqrt{\mu}} (\mu - y^2) = 2y\big|_{\pm\sqrt{\mu}} = \mp 2\sqrt{\mu}.$$

That means that $y = -\sqrt{\mu}$ is an unstable fixed point and $y = \sqrt{\mu}$ is a stable fixed point. We do not have to worry about the non-centre variables because their characteristics are stable with respect to small perturbations around $(\mathbf{0}, 0)$.

We can record this graphically in what is known as a BIFURCATION DIAGRAM:




The red lines indicate the locations of the fixed points as $\mu$ varies away from 0. The dashed line indicates an unstable fixed point and a solid line indicates a stable fixed point. We call this type of bifurcation a SADDLE-NODE BIFURCATION. The statement that these transversality conditions guarantee a saddle-node bifurcation is known as Sotomayor's Theorem (see *Perko*, pg. 338).

*2. Transcritical bifurcation*

By allowing one higher order of degeneracy, we arrive at the transversality/non-degeneracy condition:

$$\left.\frac{\partial^2 g}{\partial \mu\, \partial y}\right|_{(\mathbf{x}_0,\mu_0)} \neq 0. \tag{36}$$

Integrating the equations (34) and (36) leads us now to the following equation for the centre variable:

$$\dot{y} = \mu y - y^2 + O(\mu^2, y^3),$$

where again, we have taken $\mathbf{x}_0 = \mathbf{0}$ and $\mu_0 = 0$.

Sufficiently close to $(\mathbf{x}_0, \mu_0)$, there are critical points at $y^2 - \mu y = 0$. That is, at $y = 0$ and at $y = \mu$. There are always two critical points as $\mu$ varies over $\mathbb{R}$, except at $\mu = 0$.

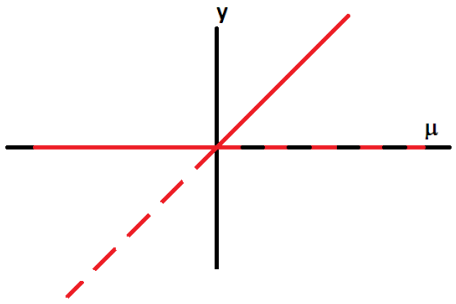
We can again analyse the stability of the critical points as $\mu$ varies away from 0 over $\mathbb{R}$. The linearized system is governed by

$$\frac{\partial}{\partial y}(\mu y - y^2) = \mu - 2y.$$

This derivative is positive or negative according as $\mu$ is positive or negative along $y = 0$. This means the fixed point $y = 0$ is unstable when $\mu > 0$, and stable when $\mu < 0$.

This derivative is negative or positive according as $\mu$ is positive or negative along $y = \mu$. This means that the fixed point $y = \mu$ is stable when $\mu > 0$, and unstable when $\mu < 0$.

We can again record this type of bifurcation graphically:




The red lines indicate the locations of the fixed points as $\mu$ varies away from 0. The dashed line indicates an unstable fixed point and a solid line indicates a stable fixed point. We call this type of bifurcation a TRANSCRITICAL BIFURCATION.

*3. Pitchfork bifurcation*

Let us continue allowing one higher order of degeneracy. We allow

$$\left.\frac{\partial^2 g}{\partial y^2}\right|_{(\mathbf{0},0)} = 0,$$

in addition to (34) but require

$$\left.\frac{\partial^3 g}{\partial y^3}\right|_{(\mathbf{0},0)} \neq 0. \tag{37}$$

These conditions then lead us as before to:

$$\dot{y} = \mu y - y^3 + O(\mu^2, y^4).$$

Sufficiently close to $(\mathbf{0}, 0)$ there are critical points at $y^3 - \mu y = 0$. That is, at $y = 0$, and at $y^2 - \mu = 0$. When $\mu < 0$, there is only one critical point. At $\mu > 0$, there are three.

The stability of $y = 0$ as $\mu$ varies away from 0 depends on

$$\left.\frac{\partial}{\partial y}\right|_{y=0} (\mu y - y^3) = \mu,$$

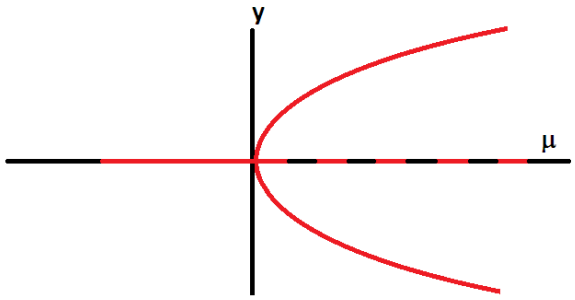
which implies stability when $\mu < 0$ and instability when $\mu > 0$.

At $y = \pm\sqrt{\mu}$, we find

$$\left.\frac{\partial}{\partial y}\right|_{y=\pm\sqrt{\mu}} (\mu y - y^3) = \mu - 3\mu = -2\mu.$$

That is, both these critical points are stable where they exist, which is only over $\mu > 0$.

This this type of bifurcation is known as the PITCHFORK BIFURCATION, and its bifurcation diagram is:




The naturality with which these bifurcations have arisen and their simplicity suggest that they arise often and in many simplified/approximate models. This is indeed the case and we shall look at some examples next time.

## 23. Lecture XXIII: One-Dimensional Local Bifurcations II

23.1. **Hopf Bifurcation.** We continue with our discussion on transversality/non-degeneracy as the origin of one dimensional/codimension one bifurcations.

*4. Hopf bifurcation*

We can think of relaxing non-degeneracy conditions as introducing symmetries. A symmetry of a system, roughly speaking,is a transformation of underlying variables that leaves the dynamics exactly unchanged. For example, $\mathbb{R}^2$, augmented with the usual Euclidean distance by which it is measured, is symmetric under rotation, reflections, and translations. If $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is any of the three types of operations aforementioned, and $\rho(\mathbf{x}, \mathbf{y})$ the distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then

$$\rho(\sigma(\mathbf{x}), \sigma(\mathbf{y})) = \rho(\mathbf{x}, \mathbf{y}).$$

If a function $V : (x, y, z) \in \mathbb{R}^3 \mapsto V(x, y, z)$ is symmetric with respect to translation in $y$, this can be expressed as

$$\frac{\partial V}{\partial y} = 0;$$

that is, $V$ is left unchanged under translation in $y$. If this condition were restricted to a particular point — a local optimum — we can think of this condition as an infinitesimal symmetry in a vanishingly small neighbourhood of that point. Symmetry can have the effect of collapsing higher dimensional problems into lower-dimensional ones. For example, a problem with rotational symmetry in $d$ dimensions is essentially a one-dimensional problem because the angular coordinates are all symmetrical and only dynamics along the radial coordinates matter.

This brings us to the symmetry condition that we shall use to obtain another type of bifurcation:

Instead of requiring that $\mathrm{D}f(\mathbf{x}_0, \mu_0)$ from (33) in Lecture XXII have only one zero eigenvalue, let us suppose that $\mathrm{D}f$ has one and only pair of purely imaginary eigenvalues. Recall that D refers to derivative in the spatial coordinates only, and not in $\mu$, so $\mathrm{D}f$ is a $(d \times d)$-matrix. We must in turn require that these eigenvalues be conjugates if $f$ is real. Let $\lambda_+ = \lambda_+(\mu_0)$ and $\lambda_- = \bar{\lambda}_+$. We shall restrict this symmetry with the non-degeneracy condition:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\mu}\right|_{\mu=0} \Re(\lambda) \neq 0, \tag{38}$$

where we take the critical point to be at $\mu_0 = 0$. This condition replaces (34).

Putting the "centre" part of the $(d \times d)$-system $\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$ in a normal form, we know that two conjugate imaginary eigenvalues mean that the centre variables $x$ and $y$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(x^2, y^2, xy, \mu^2)$$

(If we append to this system the equation $\dot{\mu} = 0$ for the parameter $\mu$ as well, we have a three-dimensional centre manifold for the $(d+1)$-dimensional system.) The eigenvalues of the linearization are $\lambda_\pm(\mu) = \pm i\omega + \mu$, which reduce to $\lambda_\pm(0) = \pm i\omega$ at $\mu = 0$.

It turns out that for systems of dimension $d > 2$ with two conjugate purely imaginary eigenvalues at $\mu = 0$, with the non-degeneracy condition (38), more can be shown about higher-order terms:

(i) at the critical point $(\mathbf{0}, 0)$, the centre manifold to the $(d \times d)$-system is two dimensional, and the system on the centre manifold can be put into the normal form:

$$\dot{x} = -y + ax(x^2 + y^2) - by(x^2 + y^2) + O(|\mathbf{x}|^4)$$
$$\dot{y} = x + bx(x^2 + y^2) + ay(x^2 + y^2) + O(|\mathbf{x}|^4),$$

and

(ii) any sufficiently smooth system $\dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu})$ with $\nu \in \mathbb{R}^m$ being a vector-valued parameter (or, are $m$ independent scalar parameters) for which $\dot{x} = g(\mathbf{x}, \mathbf{0})$ is topologically equivalent to $\dot{x} = f(\mathbf{x}, 0)$ takes the following form on the local centre manifold:

$$\begin{aligned} \dot{x} &= \mu x - y + ax(x^2 + y^2) - by(x^2 + y^2) \\ \dot{y} &= x + \mu y + bx(x^2 + y^2) + ax(x^2 + y^2). \end{aligned} \tag{39}$$

Take a moment to appreciate what this means — this means that *any* system that looks like $\dot{x} = f(\mathbf{x}, 0)$ at the origin/critical point has the same centre manifold behaviour, as long as this system governed by $f$ satisfies the non-degeneracy condition (38) for its only two purely imaginary eigenvalues.

A system $\dot{x} = g(\mathbf{x}, \boldsymbol{\nu})$ for which $\dot{x} = g(\mathbf{x}, \mathbf{0})$ is topologically equivalent to $\dot{x} = f(\mathbf{x}, 0)$ is known as an UNFOLDING of the system $\dot{x} = f(\mathbf{x}, 0)$. We think of the role of the parameters $\boldsymbol{\nu}$ as unpacking all possible degenerate behaviour that can be hidden in $\dot{x} = f(\mathbf{x}, 0)$, or otherwise that the system $\dot{x} = f(\mathbf{x}, 0)$ sits in the intersection of all its possible unfoldings. Now there are some systems with the fascinating property that it takes only finitely many parameters to characterise all possible unfoldings.

For systems with only two eigenvalues with zero real parts, and a non-degeneracy condition given by (38), for example, we see that all unfoldings look locally alike, and one parameter is enough to describe them. We call this fullest possible unfolding of $\dot{x} = f(\mathbf{x}, 0)$ its UNIVERSAL UNFOLDING.

Let us take a closer look at the centre manifold dynamics of the universal unfolding. In polar coordinates, (39) becomes:

$$\dot{r} = \mu r - ar^3 \tag{40}$$

$$\dot{\vartheta} = -1 - br^2. \tag{41}$$

We can normalize the radial equation to

$$\dot{r} = \mu r - r^3.$$

This of course exhibits a pitchfork-like bifurcation if $a \neq 0$. But the interpretation of this variable as a radial variable yields visually very different phase portrait changes as $\mu$ passes through 0, and this bifurcation is known as the HOPF BIFURCATION. In particular, we require $r > 0$. There is no trajectory with $r < 0$. This bifurcation is the sudden appearance of a periodic orbit from a focus.

We in fact have the following situation: If $a > 0$, then the bifurcation exists for $\mu > 0$, the periodic solutions are stable, and the bifurcation is said to be SUPERCRITICAL. If $a < 0$, then solutions exist for $\mu < 0$, and the periodic solutions are unstable, and the bifurcation is said to be subcritical.

The situation for $d = 2$ can be made clearer. The existence of a Hopf bifurcation requires in place of the eigenvalue condition a "genericity condition",

$$a \neq 0.$$

This arises from a Lyapunov condition on the Poincaré map, and $a$ can be calculated from a formula. For this we refer to Theorem 1 of pg 352 in *Perko*.

The Van der Pol system exhibits a Hopf bifurcation. We leave casting the Van er Pol equations into the form (39) and calculating the Lyapunov number as an exercise.

### 23.2. **Examples of codimension one bifurcations.**

**Example 23.1** (Augmented Lotka-Volterra model)**.** Recall the augmented Lotka-Volterra model of Lecture 7 (Example 7.4) for predator-prey dynamics:

$$\begin{aligned} \dot{x} &= x\left(\frac{x-\varepsilon}{x+\varepsilon}\right)\left(1 - \frac{x}{K}\right) - xy \\ \dot{y} &= \rho(xy - y). \end{aligned}$$

Recall that $x(t)$ modelled the prey population and $y(t)$ the predator population, that $K$ is the carrying capacity, $\rho$ the death rate for predators, and $\varepsilon$ the self-sustaining population parameter, which is $< 1$. For simplicity, let us take $\varepsilon = 0$.

Recall also that the equilibria are at $(0,0)$ — the extinction equilibrium, at $(K,0)$ — the prey only equilibirum, and at $(1, 1-1/K)$, the coexistence equilibrium.

Taking $K$ to be the bifurcation parameter, we find that the the coexistence equilibrium and the extinction equilibrium coincides as $K$ varies through 1, and bifurcates from it on either side of $K = 1$. On inspection of the first equation

$$\dot{x} = x(1 - x/K) - xy,$$

we can readily see that this is a transcritical bifurcation.

**Example 23.2** (Activator-Inhibitor system)**.** Recall that in Example 7.5, we considered activator-inhibitor systems modelling the concentration of two reacting species of chemicals. We also considered this system at the beginning of Lecture 22.

The activator-inhibitor system is given by the equations:

$$\dot{x} = \sigma \frac{x^2}{1+y} - x$$
$$\dot{y} = \rho(x^2 - y).$$

The parameters $\sigma$ and $\rho$ are positive.

The fixed points are:

$$(x,y) \in \{(0,0), (r_+, r_+^2), (r_-, r_-^2)\},$$

where

$$y = \sigma x - 1, \quad y = x^2 \qquad \Longrightarrow \qquad r_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}.$$

Therefore the fixed points not at zero only exist in the phase space when $\sigma > 2$.

Discounting the trivial equilibrium in which there are no chemicals of either species at all, as $\sigma$ decreases to $\sigma = 2$, the system undergoes a saddle-node bifurcation.

When $\sigma > 2$, the equilibrium $(r_-, r_-^2)$ is always a saddle.

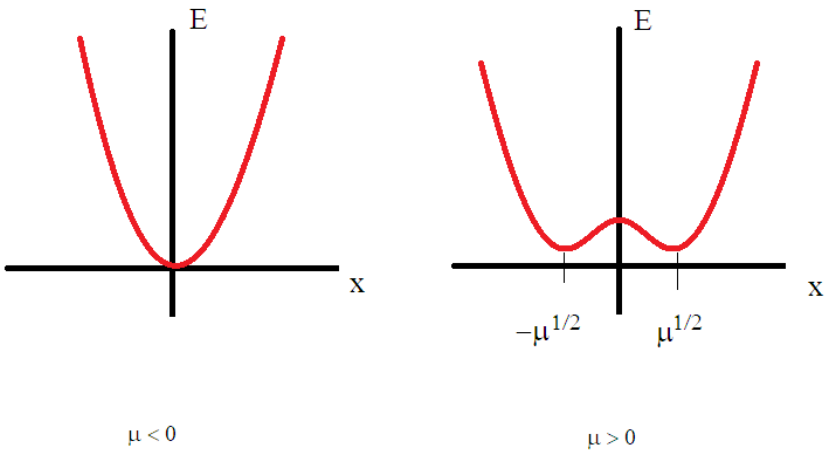
When $\sigma > 2$, the equilibrium $(r_+, r_+^2)$ is always a node. It is a source or a sink according as $\rho < 1$ or $\rho > 1$. If $\rho < 1$, then there is in fact little change in behaviour through the bifurcation point — essentially all solutions decay to nought. But if $\rho > 1$, we see that that $(r_+, r_+^2)$ is an attracting fixed point — a chemical equilibrium between two reacting species that suddenly fails to become a chemical equilibrium at $\sigma = 2$.

What we have seen is in fact a higher dimensional bifurcation with respect to the parameter $\rho$ at $\rho = 1$, but we shall not pursue this discussion now.

**Example 23.3.** This is not so much an example as an interpretation of a pitchfork bifurcation. Recall that a pitchfork bifurcation is given in normal form by

$$\dot{x} = \mu x - x^3.$$

In fact pitchfork bifurcations, like their close cousins the Hopf bifurcation, also exhibit criticality behaviour. In the equation above, as we have seen, as $\mu$ increases through $\mu = 0$, a stable fixed point bifurcates into three — an unstable fixed point at $x = 0$, and two stable fixed points at $x = \pm\sqrt{\mu}$. This can be found in physical systems where an energy potential well changes into two potential wells:




This sort of bifurcation is known as a SUPERCRITICAL, FORWARD, or SAFE pitchfork bifurcation.

Alternatively, if the sign in front of $x^3$ were plus instead of minus, we see that as $\mu$ *decreases* through $\mu = 0$, an unstable fixed point bifurcates into three — a stable fixed point at $x = 0$, and two unstable fixed points at $x = \pm\sqrt{-\mu}$. This corresponds to a reflection of the potential wells above about a horizontal.

This sort of bifurcation is known as a SUBCRITICAL, BACKWARDS, INVERTED, or UNSAFE pitchfork bifurcation.

## 24. Lecture XXIV: One-Dimensional Global Bifurcations I

24.1. **Limit cycles of systems dependent on a parameter.** Just as we discussed linearizations about periodic orbits after analogous discussions about linearizations about critical points, we shall now discuss bifurcations at periodic orbits.

Recall that when there is a periodic orbit, one eigenvalue of $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0)$ is always equal to 1, for $\mathbf{x}_0$ a point on that periodic orbit. With reference to the stable manifold theorem for periodic orbits (Thm. 16.3) periodic orbit $\Gamma$ is nonhyperbolic when there is at least another eigenvalue on the complex unit circle.

There are three ways this can occur. Either $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0)$ has another eigenvalue of 1, or it has $-1$ for an eigenvalue, or it has a complex conjugate pair $e^{\pm i\vartheta}$ of eigenvalues.

Let us consider again the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$$

on $\mathbb{R}^d$, and suppose that at $\mu = 0$, this system has a periodic orbit given by $\gamma_0(t) = \phi_t(\mathbf{x}_0, 0)$. Let $\Sigma$ be a hyperplane through $\mathbf{x}_0$ perpendicular to $\gamma_0'(0)$. To each $\mu$ in a small neighbourhood of $\mu$, and $\mathbf{x} \in \Sigma$ in a small enough neighbourhood of $\mathbf{x}_0$, we can define a first return time $\tau(\mathbf{x}, \mu)$. This is sufficient to define a Poincaré map

$$\boldsymbol{\Pi} : (\mathbf{x}, \mu) \mapsto \phi_{\tau(\mathbf{x},\mu)}(\mathbf{x}, \mu).$$

In this lecture we shall be considering bifurcations about non-hyperbolic periodic orbits. This means that we require that at the bifurcation point $\mu_0$,

$$\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) = \mathbf{x}_0, \qquad \mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) = 1,$$

already, analogous to the conditions presented by the centre manifold theorem in (34).

Recall from Thm.16.1 that along a periodic orbit $\gamma_\mu(t)$ of a planar $C^1$-system of period $T_\mu$, the Poincare map is given by

$$\mathrm{D}\boldsymbol{\Pi}(\gamma_\mu(0)) = \exp\Big(\int_0^{T_\mu} (\nabla \cdot f)(\gamma_\mu(t), \mu)\, \mathrm{d}t\Big).$$



We shall compute $\mathrm{D}\boldsymbol{\Pi}$ for simple examples and see how it characterizes bifurcations. Similar calculations can be made to find the sign on $a$ in (39) in our discussion on Hopf bifurcation in Lecture 23.

Another way to test for stability, in polar coordinates, is see if $\dot{r}$ has a definite sign for $r$ slightly beyond the limit cycle, and for $r$ slightly below the limit cycle. This technique is slightly more ad-hoc, and works best for limit cycles that are circles. Even if the system were written in polar coordinates, we can test the stability of limit cycles by looking at the Poincaré map using the polar coordinate transformations

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix} = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ -\sin(\vartheta) & \cos(\vartheta)/r \end{pmatrix} \begin{pmatrix} \partial/\partial r \\ \partial/\partial \vartheta \end{pmatrix},$$

and

$$\dot{x} = \dot{r}\cos(\vartheta) - r\sin(\vartheta)\dot{\vartheta}, \qquad \dot{y} = \dot{r}\sin(\vartheta) + r\cos(\vartheta)\dot{\vartheta},$$

so that

$$(\nabla \cdot f)((r, \vartheta), \mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r}.$$

The following examples are taken directly from section 4.5 of *Perko* and Cap. 9 of *Cain and Shaeffer*.

24.2. **Examples of simple bifurcations about nonhyperbolic periodic orbits.**

*1. Saddle-node bifurcation*

**Example 24.1.** Consider the system

$$\dot{r} = (r-1)^2 + \mu$$
$$\dot{\vartheta} = 1.$$

We find limit cycles at

$$r_\pm(\mu) = 1 \pm \sqrt{-\mu}$$

for $-1 < \mu < 0$. At $\mu \geq 0$, there is only one limit cycle, and at $\mu < -1$, there is again only one limit cycle because $r$ is a non-negative quantity.

We can test the stability of the limit cycles in the range $-1 < \mu < 0$. At $r_\pm(\mu) + \varepsilon$, we find that

$$\dot{r} = (r_\pm(\mu) + \varepsilon - 1)^2 + \mu = (\varepsilon \pm \sqrt{-\mu})^2 + \mu = \varepsilon^2 \pm 2\varepsilon\sqrt{-\mu}.$$

Choosing $\varepsilon$ small enough so that $0 < \varepsilon < \sqrt{-\mu}$, we find that $\dot{r}$ is positive at $r_+(\mu) + \varepsilon$ and negative at $r_-(\mu) + \varepsilon$. Choosing $\varepsilon$ small enough (in magnitude) so that $-\sqrt{-\mu} < \varepsilon < 0$, we find that $\dot{r}$ is negative at $r_+(\mu) + \varepsilon$ and positive at $r_-(\mu) + \varepsilon$.

This means that the limit cycle at $r_+(\mu)$ is unstable and the limit cycle at $r_-(\mu)$ is stable. Therefore the bifurcation at $\mu = 0$ is a saddle-node bifurcation. Moreover, at $\mu = -1$, we see a "reflected" Hopf-like bifurcation.

**Example 24.2.** Next look at the system

$$\dot{x} = -y - x\big(\mu - (r^2-1)^2\big)$$
$$\dot{y} = x - y\big(\mu - (r^2-1)^2\big)$$

This is transformed into

$$\dot{r} = -r\big(\mu - (r^2-1)^2\big), \qquad \dot{\vartheta} = 1.$$

From this it is clear that there are limit cycles at

$$r_\pm(\mu) = \sqrt{1 \pm \mu^{1/2}}.$$

for $0 < \mu < 1$.

Let us test for stability with the Poincaré map this time. We can find the Poincaré map derivative using

$$(\nabla \cdot f)((r_\pm(\mu), \vartheta), \mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r} = 4(1 \pm \mu^{1/2})(\pm\mu^{1/2}).$$

Therefore

$$\mathrm{D}\boldsymbol{\Pi}((r_\pm(\mu), 0), \mu) = \exp\big(\pm 8\pi(1 \pm \mu^{1/2})\mu^{1/2}\big),$$

For $0 < \mu < 1$, we find that

$$\mathrm{D}\boldsymbol{\Pi}((r_+(\mu), 0), \mu) > 1, \qquad \mathrm{D}\boldsymbol{\Pi}((r_-(\mu), 0), \mu) < 1.$$

Therefore the larger limit cycle is unstable and the smaller one is stable. At $\mu = 0$, we have $\mathrm{D}\boldsymbol{\Pi}((1,0),0) = 1$, and so we see that a saddle-node bifurcation happens at the non-hyperbolic periodic orbit of the system at $\mu = 0$.

What we see here is that in fact, we have a nondegeneracy given by

$$\frac{\partial}{\partial \mu}\boldsymbol{\Pi}(\mathbf{x}_0, 0) \neq 0,$$

as a generator of this saddle-node bifurcation, for $\mathbf{x}_0$ on the periodic orbit at $\mu = 0$.

Both the saddle-node bifurcations above can be summed up graphically using bifurcation diagrams where $\mu$ is plotted against $r$.

*2. Transcritical bifurcation*

We shall see that we can likewise derive a transcritical bifurcation with a nondegeneracy condition on $\mathbf{\Pi}$ analogous to the one we had on the "centre equation" in Lecture 22 in (36).

**Example 24.3.** Let us inspect this time the system

$$\dot{x} = -y - x(1-r^2)(1+\mu-r^2)$$
$$\dot{y} = x - y(1-r^2)(1+\mu-r^2).$$

In polar coordinates, we find

$$\dot{r} = -r(1-r^2)(1+\mu-r^2), \qquad \dot{\vartheta} = 1.$$

Again, it is clear that we have limit cycles at

$$r_+(\mu) = 1, \qquad r_-(\mu) = \sqrt{1+\mu}.$$

for all values of $\mu > -1$.

The divergence of the flux is

$$(\nabla \cdot f)((r_\pm(\mu), \vartheta), \mu) = \frac{\partial \dot{r}}{\partial r} + \frac{\partial \dot{\vartheta}}{\partial \vartheta} + \frac{\dot{r}}{r} = \begin{cases} 2\mu & r = r_+(\mu) \\ -2\mu(1+\mu) & r = r_-(\mu) \end{cases}.$$

So as $\mu$ increases through 0, the orbit at $r_+$ changes from being stable to being unstable, and the orbit at $r_-(\mu)$ changes from being stable to being unstable. This is characteristically a transcritical bifurcation.

Calculating the actual derivative of the Poincaré map, we find

$$\mathrm{D}\mathbf{\Pi}((r_\pm(\mu), 0), \mu) = \begin{cases} e^{4\pi\mu} & r = r_+(\mu) \\ e^{-4\pi\mu(1+\mu)} & r = r_-(\mu) \end{cases}.$$



And we see that the non-degeneracy condition that would have induced this bifurcation is

$$\frac{\partial}{\partial \mu}\mathrm{D}\mathbf{\Pi}(\mathbf{x}_0, 0) \neq 0, \tag{42}$$

for $\mathbf{x}_0$ on the periodic orbit of the system at $\mu = 0$.

Again, it is possible to represent this bifurcation graphically by plotting $\mu$ against $r$.

*3. Pitchfork bifurcation*

**Example 24.4.** Finally let us look at the following system:

$$\dot{x} = -y + x(1-r^2)\big(\mu - (r^2-1)^2\big)$$
$$\dot{y} = x + y(1-r^2)\big(\mu - (r^2-1)^2\big).$$

In polar coordinates, this becomes

$$\dot{r} = r(1-r^2)\big(\mu - (r^2-1)^2\big)$$
$$\dot{\vartheta} = 1.$$

Again limit cycles are circles, and of radii

$$r_0(\mu) = 1, \qquad r_\pm(\mu) = \sqrt{1 \pm \mu^{1/2}}.$$

There are three limit cycles where $0 < \mu < 1$. As before, we can compute the derivative of the Poincaré map to test for stability, and we find that

$$\mathrm{D}\boldsymbol{\Pi}((r_0(\mu), 0), \mu) = e^{-4\pi\mu}, \qquad \mathrm{D}\boldsymbol{\Pi}((r_\pm(\mu), 0), \mu) = e^{4\pi\mu\left(1 \pm \mu^{1/2}\right)}.$$

This is a (reflected) subcritical pitchfork bifurcation.

24.3. **Degeneracy conditions for bifurcations.** We can sum up and fill in the various degeneracy conditions for simple bifurcations about a nonhyperbolic periodic orbit in a theorem:

**Theorem 24.1.** *Let $\Gamma$ be a nonhyperbolic periodic orbit of a $C^2$-planar system at $\mu = \mu_0$ containing the point $\mathbf{x}_0$. Let $\boldsymbol{\Pi}$ be the Poincaré map defined in a neighbourhood of $(\mathbf{x}_0, \mu_0)$ for the orbit $\Gamma_0$.*

*The non-degeneracy conditions*

$$\mathrm{D}^2\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \qquad \frac{\partial}{\partial \mu}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a saddle-node bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*

*The relaxed non-degeneracy conditions*

$$\frac{\partial}{\partial \mu}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) = 0, \qquad \frac{\partial}{\partial \mu}\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \qquad \mathrm{D}^2\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a transcritical bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*

*The further relaxed non-degeneracy conditions*

$$\frac{\partial}{\partial \mu}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) = 0, \quad \mathrm{D}^2\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) = 0, \quad \frac{\partial}{\partial \mu}\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0, \quad \mathrm{D}^3\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0) \neq 0$$

*induces a pitchfork bifurcation about $\Gamma_0$ at $\mu = \mu_0$.*

## 25. Lecture XXV: One-Dimensional Global Bifurcations II

Last lecture, we established that there are three ways that a periodic orbit $\Gamma_{\mu_0}$ can be nonhyperbolic. For a point $\mathbf{x}_0 \in \Gamma_{\mu_0}$, since $\Gamma_{\mu_0}$ is periodic, there is always an eigenvalue of the derivative of its Poincaré map, $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0)$, that takes the value 1. Nonhyperbolicity then requires that either $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0)$ has another eigenvalue of 1, or it has a complex conjugate pair $e^{\pm i\vartheta}$ of eigenvalues, or it has the eigenvalue $-1$. We had only considered the first case last time. In this lecture we shall take a closer look at the remaining two cases.

25.1. **Neimark-Sacker bifurcation.** In this subsection we shall consider bifurcations about a periodic orbit where $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0)$ has a pair of conjugate eigenvalues $e^{\pm i\vartheta}$ in addition to the eigenvalue 1. In order to have three eigenvalues, the dynamics must take place in at least three spatial dimensions. Therefore, this sort of bifurcation does not occur on the plane. We shall see that with the correct non-degeneracy conditions, this sort of spectral (eigenvalues) behaviour leads to the periodic orbit analogue of the Hopf bifurcation (which occurs about a critical point). It is again simplest to look at an example:

**Example 25.1.** Consider the system:

$$\begin{aligned}\dot{r} &= \mu(r-1) - \omega\rho - (r-1)\sqrt{(r-1)^2+\rho^2}\\ \dot{z} &= \omega(r-1) + \mu\rho - \rho\sqrt{(r-1)^2+\rho^2}\\ \dot{\vartheta} &= 1.\end{aligned}$$

For any $\mu$, we have the periodic orbit

$$r(t) = 1, \qquad \rho(t) = 0, \qquad \vartheta(t) = t,$$

where the angular variable $\vartheta$ really only needs to be defined up to multiples of $2\pi$. It is evident that $\dot{r} = \mathrm{d}(r-1)/\mathrm{d}t$. We use $r$ instead of $r-1$ as a variable here in order to interpret this stationary solution as a periodic orbit in $\mathbb{R}^3$, where the $xy$-plane is written in polar coordinates, instead of as a critical point. The final argument of the cylindrical coordinate system is $z$, the height from the the $xy$-plane.



Let $\mathbf{x}_0$ be a point on the orbit. The Poincaré map is the first return map taken at a particular angular slice, and hence ignores the angular $\vartheta$ coordinate. The linearization about the periodic orbit gives

$$\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu) = \exp(2\pi\mathbf{C}_\mu), \qquad \mathbf{C}_\mu := \begin{pmatrix}\mu & -\omega\\ \omega & \mu\end{pmatrix}$$

This gives us eigenvalues of

$$e^{2\pi\mu \pm 2\pi i\omega},$$

with a bifurcation at $\mu_0 = 0$.

To see the sort of dynamics occuring, we see from the eigenvalues that for $\mu < 0$, any nearby solutions decay to $(r, z) = (1, 0)$.

For $\mu > 0$, the periodic orbit is no longer stable. However, there is an invariant manifold in the form of a torus, of major raidus 1 and minor radius $\sqrt{\mu}$ to which solutions starting near the periodic orbit converges. That is, we see that for any arbitrary phase $\phi$ depending on initial conditions, we have the limit cycles

$$r(t) = 1 + \sqrt{\mu}\cos(\omega t + \phi), \quad z(t) = \sqrt{\mu}\sin(\omega t + \phi), \quad \vartheta = t,$$

and these limit cycles fill up the the surface of a torus of the dimensions described.

If we collapse the angular variable $\vartheta$, which we can do without danger of trajectories crossing paths because the remaining two equations are still an autonomous system, we find a Hopf-like bifurcation for the planar system with variables

$$\mathfrak{x} = r - 1, \qquad \mathfrak{y} = z.$$

Indeed, using the substitution on the first two equations of the system, we find

$$\dot{\mathfrak{x}} = \mu\mathfrak{x} - \omega\mathfrak{y} - \mathfrak{x}\sqrt{\mathfrak{x}^2+\mathfrak{y}^2}$$
$$\dot{\mathfrak{y}} = \omega\mathfrak{x} + \mu\mathfrak{y} - \mathfrak{y}\sqrt{\mathfrak{x}^2+\mathfrak{y}^2}.$$

Now using $R^2 = \mathfrak{x}^2 + \mathfrak{y}^2$ on the $\mathfrak{x}\mathfrak{y}$-plane, we find

$$2\dot{R} = \mu R - R^2,$$

which looks like a slightly scaled transcritical bifurcation (cf. Lecture 22.1.2), but interpreted as a radial variable on a plane (i.e., $R \geq 0$), has a bifurcation diagram that coincides with the Hopf bifurcation. (When the lower half-plane is ignored, as in the radial case, a pitchfork bifurcation and a transcritical bifurcation looks the same.)

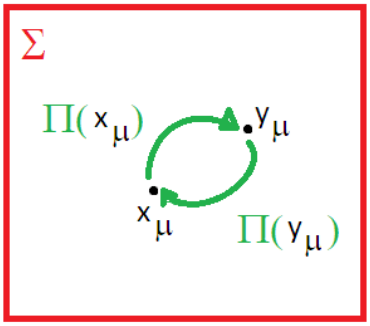
25.2. **Period-doubling bifurcation.** On the plane, we have not seen an eigenvalue of $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0)$ taking the value $-1$, either. It is, in fact, again, impossible. Let us briefly explore what an eigenvalue of $-1$ signifies.

Recall that $\mathrm{D}\boldsymbol{\Pi}(\mathbf{x}_0, \mu_0)$ always has one eigenvalue that takes the value 1 because that is the multiplier associated with the direction normal to the hyperplane $\Sigma$ on which $\boldsymbol{\Pi}$ is defined. An eigenvalue of $-1$ therefore means that along some direction inside/along the hyperplane, each return of a trajectory to $\Sigma$ occurs in an opposite sense from the previous one. In particular, there are trajectories $\Gamma_\mu$ close to our nonhyperbolic periodic orbit, which intersect $\Sigma$, say, at $\mathbf{x}_\mu$, returns at $\mathbf{y}_\mu \in \Sigma$ (along the same direction transverse to $\Sigma$) and then returns again to complete itself as a periodic orbit, intersecting $\Sigma$ the second time at $\mathbf{x}_\mu$. The direction associated with the eigenvalue $-1$ is then $\mathbf{x}_\mu - \mathbf{y}_\mu$, which lies on $\Sigma$, because the next time it returns, i.e., on a second application of the Poincaré map, the point is mapped along this vector in the opposite direction at the same magnitude — thereby taking $\mathbf{y}_\mu$ back to $\mathbf{x}_\mu$ in the second return, and closing up a periodic orbit with *approximately* twice the period — that is,

$$\boldsymbol{\Pi} \circ \boldsymbol{\Pi}(\mathbf{x}_\mu, \mu) = \mathbf{x}_\mu$$

(see also Figure 6. on pg 371 of *Perko*):



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This cannot happen for plane autonomous systems because a trajectory will have to cross itself in order for a period doubling phenomenon to occur. One way to see this is, supposing such a trajectory existed on the plane, arbitrarily pick a centre $\mathcal{O}$ of the period doubling trajectory. Either $|\mathbf{x}_\mu - \mathcal{O}| < |\mathbf{y}_\mu - \mathcal{O}|$, or vice versa. As we continue along the trajectory and make our way back from $\mathbf{y}_\mu$ to $\mathbf{x}_\mu$, arbitrarily close to $\mathbf{x}_\mu$, the radius will have had to have decreased (resp. increased) back to $|\mathbf{x}_\mu - \mathcal{O}|$, which inplies a crossing.

This sort of behaviour clearly depends on the existence of periodic orbits that bend very close back on themselves at $\mu$ close to $\mu_0$, very much like in noodle dough that is folded and stretched

multiple times to make noodle. It is a behaviour closely associated with what is normally called chaotic behaviour, and can become very complicated.

**Example 25.2.** Let us consider a toy example of a system with D$\mathbf{\Pi}$ having an eigenvalue of $-1$:

$$\begin{aligned}\dot r &= -(r-1) - \frac{1}{2}z + \mu\big(\cos^2(\vartheta/2)(r-1) + \sin(\vartheta/2)\cos(\vartheta/2)z\big) - (r-1)\sqrt{(r-1)^2+z^2}\\ \dot z &= \frac{1}{2}(r-1) - z + \mu\big(\sin^2(\vartheta/2)z + \sin(\vartheta/2)\cos(\vartheta/2)(r-1)\big) - z\sqrt{(r-1)^2+z^2}\\ \dot\vartheta &= 1.\end{aligned}$$

Since the first two equations depend on $\vartheta$, we can no longer collapse the final equation as we did for Example 25.1 and still retain an autonomous system. Substituting $\vartheta = t$ into the remaining two equations, we can build a Poincaré map on the slice $\vartheta = 0$.

The linearization of the non-autonomous system is

$$\begin{aligned}\dot r &= -(r-1) - \frac{1}{2}z + \mu\big(\cos^2(t/2)(r-1) + \sin(t/2)\cos(t/2)z\big)\\ \dot z &= \frac{1}{2}(r-1) - z + \mu\big(\sin^2(t/2)z + \sin(t/2)\cos(t/2)(r-1)\big).\end{aligned}$$

Solutions are spanned by

$$\begin{pmatrix} r(t)\\ z(t)\end{pmatrix} \in \operatorname{span}\left\{ e^{(\mu-1)t}\begin{pmatrix}\cos(t/2)\\ \sin(t/2)\end{pmatrix}, e^{-t}\begin{pmatrix}-\sin(t/2)\\ \cos(t/2)\end{pmatrix}\right\}.$$

On first return, we have $\mathbf{\Pi}(r(0), z(0); \mu) = (r(2\pi), z(2\pi))$. Therefore,

$$\mathrm{D}\mathbf{\Pi}\begin{pmatrix} r(0)\\ 0\end{pmatrix} = -e^{2\pi(\mu-1)}\begin{pmatrix} r(0)\\ 0\end{pmatrix}, \qquad \mathrm{D}\mathbf{\Pi}\begin{pmatrix} 0\\ z(0)\end{pmatrix} = -e^{-2\pi}\begin{pmatrix} 0\\ z(0)\end{pmatrix},$$

and the eigenvalue $-e^{2\pi(\mu-1)}$ takes the value $-1$ at the bifurcation point $\mu = \mu_0 = 1$.

This example is slighly artificial, however, and by inspection, it can be checked that it has similar solutions for $\mu > 1$ to Example 25.1. It often happens that period-doubling occur at a sequence of values $\{\mu_i\}_{i=0}^\infty$ for the bifurcation parameter $\mu$, so that trajectories are, to use the noodle imagery, stretched and folded repeatedly, increasingly often at each successive bifurcation value. This phenomenon is known as the period-doubling cascade and we shall inspect it presently.



25.3. **Period doubling cascades.** Most results within our reach concerning period doubling cascades come from numerical experiments. In this subsection we shall describe two numerical examples of period doubling cascades.

**Example 25.3** (Rössler's equation)**.** A simplified Lorenz system already exhibits period doubling cascades. The Rössler system is

$$\begin{aligned}\dot x &= -y - z\\ \dot y &= x + ay\\ \dot z &= b + z(x-c).\end{aligned}$$

We consider bifurcations of the system with fixed $a = b = 1/5$, as $c$ varies.

It turns out that there are periodic solutions at $c > 0.4$. There is a first period-doubling bifurcation at $c_1 \approx 2.832$. The period of periodic orbits "doubles" again at $c_2 \approx 3.837$, and there is an infinite number of bifurcation values $\{c_k\}_{k=1}^\infty$ which accumulate at $c_\infty \approx 4.3$, at each of which there is a period-doubling bifurcation. (See Figure 9.15 of *Cain and Shaeffer*.)

Whilst we have been discussing dynamical systems defined by flows of ODEs, it is far simpler to look at period doubling bifurcations in systems defined by maps. That is, we consider, for example,

only what happens on $\Sigma$, and not the entire trajectories themselves. Let there be a collection of maps, indexed by $\mu \in \mathbb{R}$:

$$\Psi(\cdot, \mu) : \Sigma \to \Sigma.$$

This is more general than systems defined by flows, because, for example, we can take $\Sigma$ to be an interval in $\mathbb{R}$ here, even though we have shown that if $\Sigma$ is a codimension one (i.e., dimension $d-1$) surface defining a Poincaré map for a system with period doubling bifurcation, we cannot have $d - 1 = 1$.

**Example 25.4** (Logistic map)**.** Consider the archetypal period-doubling discrete system:

$$x_{n+1} = \mu x_n(1 - x_n).$$

Here $\mu \in (0, 4]$, and $x_n \in [0, 1]$. This is a discrete time version of the logistic equation used in population models.

For $\mu \in (0, 1)$, as expected $x_n \to 0$, since $x_n \leq \mu^n$.

For $\mu \in (0, 2)$, $x_n \to (\mu - 1)/\mu$, which is easy to find simply by setting $x_\infty = \mu x_\infty(1 - x_\infty)$.

For $\mu \in (2, 3)$, the same behaviour holds, but there is more oscillatory behaviour in the convergence to $(\mu - 1)/\mu$.

At $\mu \in (3, 1 + \sqrt{6})$, we see that almost all initial values lead to permanent oscillations between two different values.

At $\mu \in (1 + \sqrt{6}, \sim 3.54409)$, there is another bifurcation into oscillations between four values for almost all initial values.

As $\mu$ increases beyond $\mu_3 \approx 3.54409$, we find that there are values $\mu_4$ at which $x_n$ oscillates between eight values, then $\mu_5 > \mu_4$ at which $x_n$ oscillates between sixteen values, and so on.

At $\mu \approx 3.56995$, this period-doubling cascade stops.

We think of these as period-doubling bifurcations because in higher dimensions, the return map of a trajectory to $\Sigma$ is a discrete system.

It turns out that for all one-dimensional discrete systems $x_n = f(x_{n-1}, \mu)$ with period doubling bifurcations, where $f(x, \mu)$ has one quadratic maximum for each value of $\mu$, period-doubling cascades occur at the same rate. Where the bifurcation points are at $\{\mu_i\}_{i=0}^{\infty}$, the limit

$$\delta := \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

is universal, and is known as the FIRST FEIGENBAUM CONSTANT. The approximate value of this number is $\delta \approx 4.669201$.

## 26. Lecture XXVI: Higher Dimensional Local Bifurcations

26.1. **Universal unfoldings.** There are many different topics in dynamical systems, even as they relate to flows and ordinary differential equations, that we have not had time to cover in this module. I should like bring this set of lectures to a close by considering more closely the question of universal unfoldings. We seek to understand the totality of possible ways a system can be parameterized, and the associated bifurcations as these parameters vary. Another way of looking at this is, we seek to use extra parameters to unpack all possible near-by behaviours of a system, or, to find all collections of systems that intersect at one specified system.

Recall that a UNIVERSAL UNFOLDING of a $d$-dimensional system $\dot{\mathbf{x}} = f(\mathbf{x})$ is a system $\dot{x} = g(\mathbf{x}, \boldsymbol{\nu})$, indexed by finitely many parameters $\boldsymbol{\nu} \in \mathbb{R}^m$, such that at $\boldsymbol{\nu} = \boldsymbol{\nu}_0$, $\dot{\mathbf{x}} = g(\mathbf{x}, \boldsymbol{\nu}_0)$ is topologically equivalent to $\dot{\mathbf{x}} = f(\mathbf{x})$.

In this lecture we shall look at universal unfoldings that depend on more than one parameter. We shall limit our discussion to systems in one *spatial* dimension. The dimensionality to which the title of this lecture refers are the degrees of freedom required (by the parameters) to yield universal embeddings. This is higher dimensionality is often referred to as "bifurcations of higher *co*dimensions". We are taking another look at one dimensional bifurcations from the opposite perspective, by another organization principle, where, instead of systematically relaxing non-degeneracy conditions to get individual bifurcations, we shall systematically look at more and more degenerate/nonlinear systems and ask for all possible bifurcations at their nonhyperbolic fixed points.

First let us consider the simplest nonlinear system:

$$\dot{x} = -x^2.$$

Observe that unfolding by adding higher order terms does not change the qualitative behaviour of the system about the nonhyperbolic critical point $x = 0$. The fixed points of the system

$$\dot{x} = -x^2 + \mu_3 x^3$$

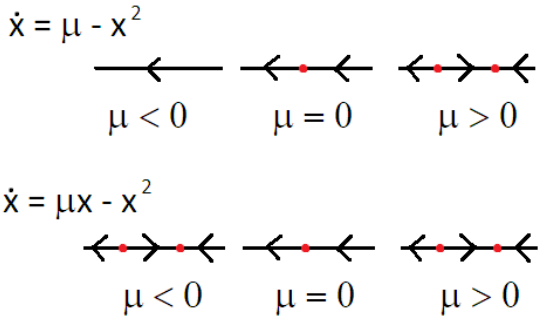
are $x = 0$ and $x = 1/\mu_3$. (We have be using $\mu_i$ to denote a sequence of bifurcation values, but here, each $\mu_i$ is not just a fixed value, but a parameter that is allowed to vary over $\mathbb{R}$.) As this unfolding approaches the original system, $\mu_3 \to 0$, and the remaining fixed point leaves any fixed neighbourhood of the nonhyperbolic fixed point. This leaves us to consider unfoldings of lower orders:

$$\dot{x} = \mu_0 + \mu_1 x - x^2$$

By an affine shift $x \mapsto (x - \mu_1/2)$, we see that this unfolding has dynamics which are topologically equivalent to

$$\dot{x} = \mu - x^2, \qquad \mu = \mu_0 - \frac{\mu_1^2}{4}.$$

This is the normal form for the saddle-node bifurcation. This verifies our earlier calculations for the saddle-node bifurcation in Lecture 22 that this normal form is the universal unfolding of the system with $f(x) = -x^2$, without non-degeneracy assumptions beyond nonhyperbolicity. In what sense is the transcritical bifurcation included in this unfolding? We need only look at the phase portraits of the normal forms expressing these two types of bifurcations to see that all behaviours/dynamics to either side of a transcritical bifurcation is included in dynamics expressed by the universal unfolding that is the normal form for a saddle-node bifurcation:




26.2. **Revisiting the pitchfork bifurcation.** Next, looking at

$$\dot{x} = -x^3,$$

we can postulate an unfolding of the form

$$\dot{x} = -x^3 + \mu_2 x^2 + \mu_1 x + \mu_0.$$

As it turns out, the normal form $\dot{x} = \mu x^2 - x^3$ for the pitchfork bifurcation is not a universal unfolding of $\dot{x} = -x^3$.

By a translation, as before, we can elimate one of the lower-order terms, so that we have the topologically equivalent system

$$\dot{x} = -x^3 + \mu_1 x + \mu_0.$$

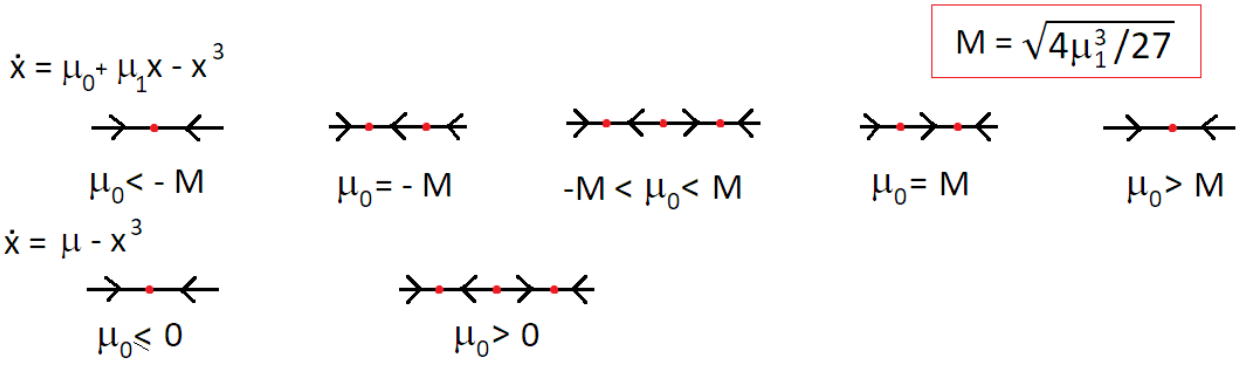
We can find the fixed points of this system as before by setting

$$0 = p(x, \mu_1, \mu_0) = -x^3 + \mu_1 x + \mu_0.$$

It is an elementary calculation to find the maxima and minima of $p$ — one minimum at $x = -\sqrt{\mu_1/3}$ and a maximum at $x = \sqrt{\mu_1/3}$. The system has three fixed points if $p$ has three roots, which is the case when $p(-\sqrt{\mu_1/3}) < 0$ and $p(\sqrt{\mu_1/3}) > 0$. This works out to be $\mu_0^2 < 4\mu_1^3/27$.

Likewise, the system has two fixed points when $\mu_1^2 = 4\mu_1^3/27$. And finally, the system has one fixed point if $\mu_0^2 > 4\mu_1^3/27$.

Its phase portrait can be compared to that of the normal form of the pitchfork bifurcation below:




So the full pitchfork bifurcation is in fact a codimension two bifurcation.

The locus of points on the $\mu_1\mu_2$-plane on which we have the phase portrait (or its reflection) not seen in the unfolding given b $\dot{x} - \mu x^3$ lies on the curve

$$\mu_0^2 = \frac{4\mu^3}{27}.$$

This sort of higher-dimensional bifurcation is known as a CUSP BIFURCATION.

This analysis for higher order systems can be continued. For $\dot{x} = -x^4$, a universal unfolding require three parameters to describe and results in a SWALLOWTAIL BIFURCATION (see Figure 2 on pg 346 of *Perko*).

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