TMA4165 2019 – exercise **9** (week 11) – *rough* solution

Problem 1

(1) $r\dot{r} = x\dot{x} + y\dot{y} = -x^4 + xy^4 - y^4 + y^5$

Expect the terms $-x^4 - y^4$ to dominate for small (x,y) so this should be asymptotically stable.

For the xy^4 term, try Young's inequality see the last page

$|xy^4| = |xy^2 \cdot y^2| \le \frac{1}{2}(xy^2)^2 + \frac{1}{2}(y^2)^2 = \frac{1}{2}x^2y^4 + \frac{1}{2}y^4$

Do it again, on the first term:

$x^2y^4 \le \frac{1}{2}x^4 + \frac{1}{2}y^8$

Put it all together:

$r\dot{r} \le -x^4 - y^4 + \frac{1}{4}x^4 + \frac{1}{4}y^8 + \frac{1}{2}y^4 + y^5$

$= -\frac{3}{4}x^4 - \frac{1}{2}y^4\left(1 - 2y^5 - \frac{1}{2}y^8\right)$

< 0 when $|y|$ is small enough.

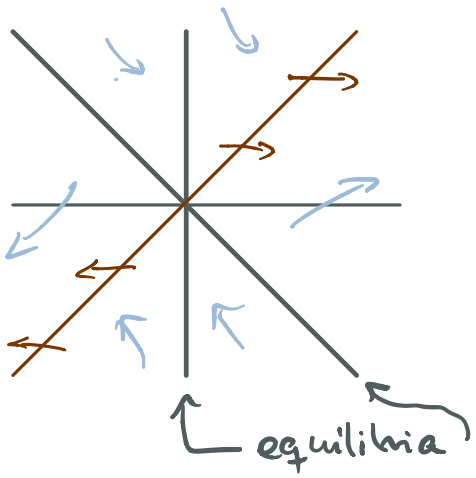
(2) $r\dot{r} = x\dot{x} + y\dot{y} = x^4 + x^3y + x^3y - xy^3$

does not look promising. Try understanding the phase diagram instead:

x-nullcline: $\dot{x} = 0 \iff x = 0$ or $x + y = 0$

y-nullcline: $\dot{y} = 0 \iff x = 0$ or $x + y = 0$ or $x - y = 0$

Hence every point with $x = 0$ or $x + y = 0$ is an equilibrium, so the origin is definitely not asymptotically stable.



The system is unchanged if we replace (x,y) by $(-x,-y)$. This symmetry helps in drawing a rough sketch of the phase diagram: ←

The picture makes us suspect that solutions in the region $x^2 \geq y^2$ will go to infinity. Why not simply try plotting $V = \frac{1}{2}(x^2 - y^2)$? We get

$$\dot{V} = x\dot{x} - y\dot{y} = x^4 + x^3y - x^3y + xy^3 = (x^3 - y^3)x$$

If $V(x,y) > 0$ then $x > |y|$ or $x < -|y|$.

In the first case, $x^3 > |y|^3 \geq y^3$, so $\dot{V} > 0$ (since $x > 0$).

In the second case, $x^3 < -|y|^3 \leq y^3$, so $\dot{V} > 0$ (since $x < 0$)

Thus we have shown that $V > 0 \Rightarrow \dot{V} > 0$,

and since every neighbourhood of $(0,0)$ contains points where $V > 0$ (and also $V(0,0) = 0$), the equilibrium at $(0,0)$ is unstable.

(4) The origin is a **saddle point** and therefore unstable. The matrix of the linearization is $\begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ with determinant $q = 3\cdot(-2) - (-2)2$

$$= -6 + 4 = -2 < 0.$$

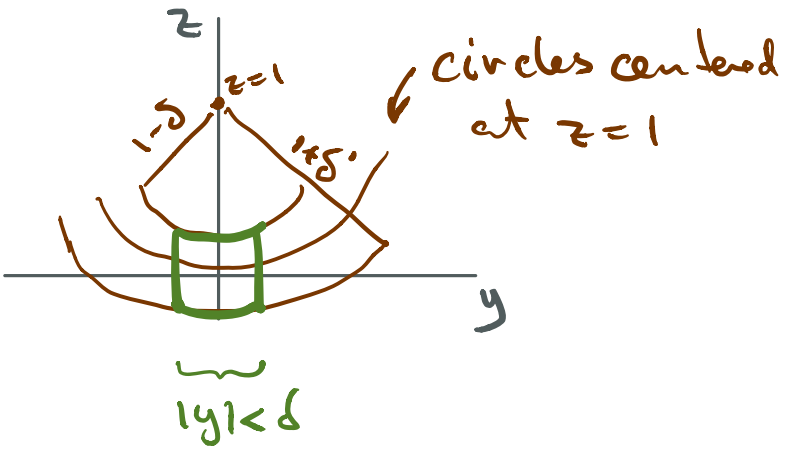
(3) From the last two equations we get $y\dot{y} + (z-1)\dot{z} = 0$, so $y^2 + (z-1)^2 = \text{constant}$ along trajectories. Similarly, the first two equations imply $x^2 + 2y^2 = \text{constant}$

So assuming $\sqrt{x(0)^2 + y(0)^2 + z(0)^2} < \delta$

we find $x^2 + 2y^2 = \text{const.} < 2\delta^2$, so $|y| < \delta$

But also $y^2 + (z-1)^2 = C$ with

$$(1-\delta)^2 < C < \delta^2 + (1+\delta)^2 \quad \text{i.e.,}$$



circles centered at $z=1$

(y,z) remains on a circle with center at $(0,1)$ and radius between $1-\delta$ and $\sqrt{1+2\delta+2\delta^2}$ (see the figure.)

We get stability, but not asymptotic.

Improved solution: Just put

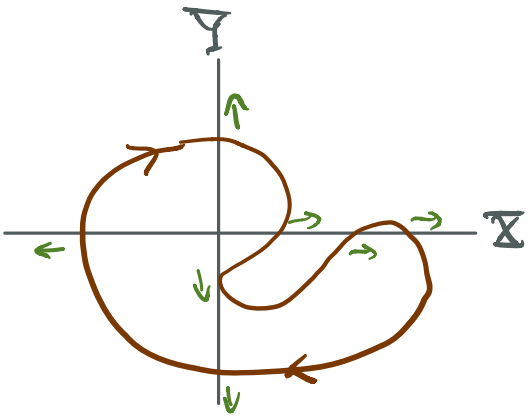
$$V(x,y,z) = x^2 + 2y^2 + \left(y^2 + (z-1)^2 - 1\right)^2$$

This function is invariant ($\dot{V} = 0$), $V(0,0,0) = 0$, and $V > 0$ elsewhere (except at $(0,0,2)$) so $(0,0,0)$ is a local minimum for V.

Thus V is a weak Lyapunov function for the system, so $(0,0,0)$ is stable.

Problem 2

The key here is to note that as (x,y) travels around the given circle, then $(X(x,y), Y(x,y))$ travels around a closed path in (X,Y)-space

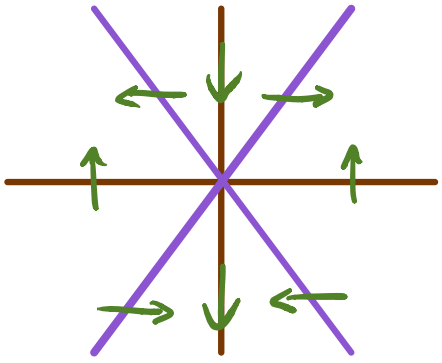


We get →↓→ from the curve touching the negative y axis without crossing it.

The example given could look like this:

We can't get ↑↓ because the path cannot get from the positive y axis to the negative y axis without crossing the x axis, and $(0,0)$ is forbidden.

Problem 3 Draw the nullclines:



$\dot{x}=0 \Leftrightarrow xy=0$ shown in brown

$\dot{y}=0 \Leftrightarrow y=\pm\sqrt{3}x$ shown in purple

Here we find $n_{\uparrow\rightarrow}=2$ and $n_{\rightarrow\uparrow}=0$

so the index is -2

Clearly, the negative y axis shows the origin is unstable:

There is a solution with $x=0$, $\dot{y}=-y^2$

namely $y=\frac{1}{t}$ for $t<0$

Phase diagram: Note the system is homogeneous, implying that a rescaled phase path is another phase path. Looking at the above picture, I expect more straight line paths: Try putting $y=Cx$.

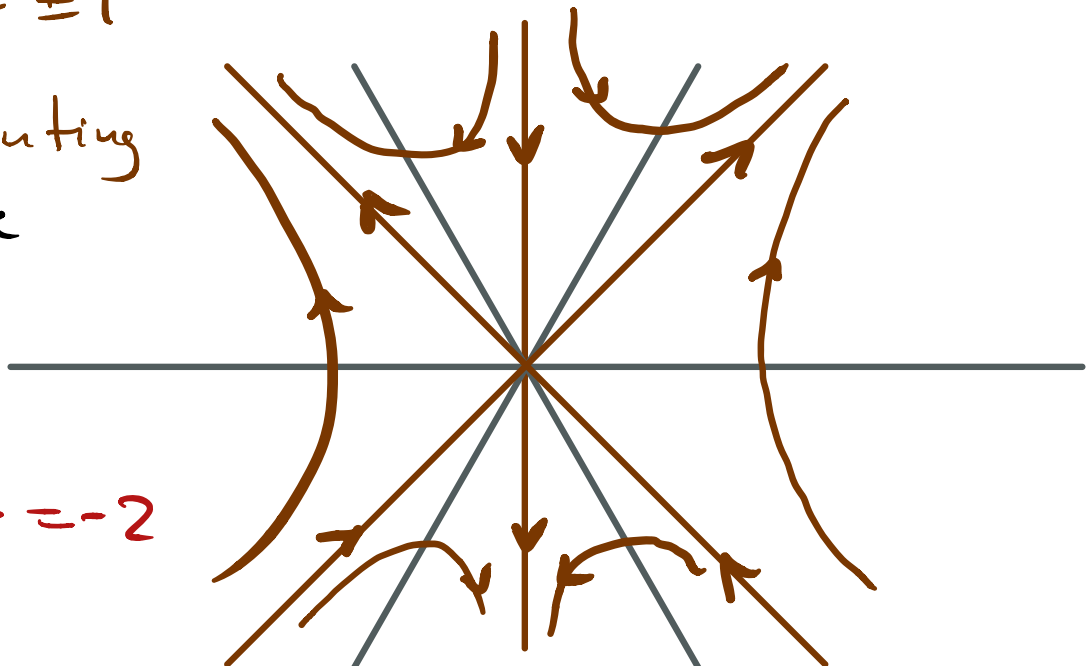
The equations become $\dot{x}=2Cx^2$, $C\dot{x}=(3-C^2)x^2$

These equations are compatible if $3-C^2=2C^2$

i.e., $C^2=\pm 1$

I am counting 6 elliptic sectors, giving

$I=1+\frac{0-6}{2}=-2$



Appendix: Young's inequality, Peter–Paul

The most elementary version of Young's inequality is derived by expanding and rearranging terms of the inequalities $(a \pm b)^2 \geq 0$, with the result

$$|ab| \leq \frac{a^2}{2} + \frac{b^2}{2}$$

The Peter-Paul inequality comes from this by replacing a by $\sqrt{2\varepsilon}|a|$ and b by $\sqrt{|b|/(2\varepsilon)}$

$$|ab| \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

The name comes from the expression "Robbing Peter to pay Paul". We make one term small at the expense of making the other term bigger.

Young's inequality has a more general form:

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q} \quad \text{where } p,q>1, \ \frac{1}{p}+\frac{1}{q}=1$$

(Proof: Use Lagrange multipliers to maximise xy on the curve $\frac{x^p}{p} + \frac{y^q}{q} = C$, $x, y > 0$. The maximum occurs where $x^p = y^q = C$.)

This also has a Peter-Paul version

$$|ab| \leq \varepsilon |a|^p + \frac{|b|^q}{q p^{q} \varepsilon^{q/p}} \quad \text{where } p,q>1, \ \frac{1}{p}+\frac{1}{q}=1$$

In practical applications, we rarely need the exact form of the second term.