Differential inequalities

Theorem. *Assume that u and v are real-valued, continuous functions on* $[0, T]$ *satisfying* $\dot{u} \le f(u,t)$ *and* $\dot{v} = f(v,t)$ *and* $u(0) \le v(0)$*, where f is continuous and Lipschitz with respect to its first variable. Then* $u(t) \le v(t)$ *for* $t \in [0, T]$.

Proof. Assume otherwise, and pick some t_1 with $u(t_1) > v(t_1)$. Let t_0 be the largest value $< t_1$ for which $u(t_0) = 0$. (It exists, by continuity and the fact that $u(0) \le v(0)$.) Then for $t_0 < t < t_1$,

$$\frac{d}{dt}(u - v) = f(u,t) - f(v,t) \le L|u - v| = L(u - v)$$

where L is the Lipschitz constant of f. It follows that $e^{-Lt}\big(u(t) - v(t)\big)$ is non-increasing on $[t-_0, t_1]$, which is a contradiction since $u(t_0) - v(t_0) = 0$ and $u(t_1) - v(t_1) > 0$. ∎