EXERCISES 9

Computations

In Exercises 1 through 6, find all orbits of the given permutation.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 6 & 2 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 4 & 8 & 3 & 1 & 7 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

4. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 1$

5. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n + 2$

6. $\sigma : \mathbb{Z} \to \mathbb{Z}$ where $\sigma(n) = n - 3$

In Exercises 7 through 9, compute the indicated product of cycles that are permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

7. $(1, 4, 5)(7, 8)(2, 5, 7)$

8. $(1, 3, 2, 7)(4, 8, 6)$

9. $(1, 2)(4, 7, 8)(2, 1)(7, 2, 8, 1, 5)$

In Exercises 10 through 12, express the permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles, and then as a product of transpositions.

10. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$

13. Recall that element a of a group G with identity element e has order $r > 0$ if $a^r = e$ and no smaller positive power of a is the identity. Consider the group S_8.

- **a.** What is the order of the cycle $(1, 4, 5, 7)$?
- **b.** State a theorem suggested by part (a).
- **c.** What is the order of $\sigma = (4, 5)(2, 3, 7)$? of $\tau = (1, 4)(3, 5, 7, 8)$?
- **d.** Find the order of each of the permutations given in Exercises 10 through 12 by looking at its decomposition into a product of disjoint cycles.
- **e.** State a theorem suggested by parts (c) and (d). [*Hint:* The important words you are looking for are *least common multiple.*]

In Exercises 14 through 18, find the maximum possible order for an element of S_n for the given value of n.

14. $n = 5$ **15.** $n = 6$ **16.** $n = 7$ **17.** $n = 10$ **18.** $n = 15$

19. Figure 9.22 shows a Cayley digraph for the alternating group A_4 using the generating set $S = \{(1, 2, 3), (1, 2)(3, 4)\}$. Continue labeling the other nine vertices with the elements of A_4, expressed as a product of disjoint cycles.

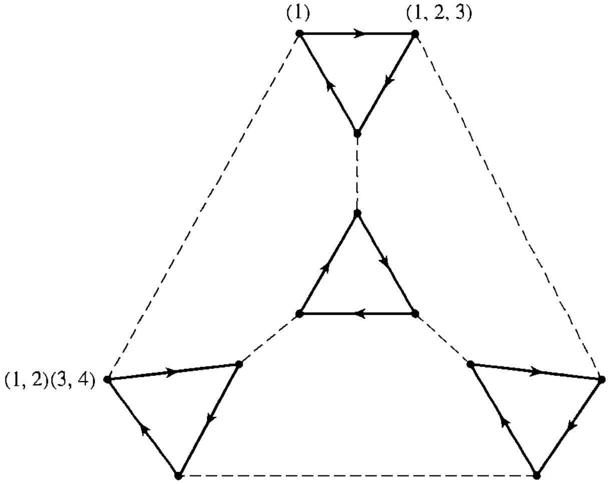
Concepts

In Exercises 20 through 22, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

20. For a permutation σ of a set A, an *orbit* of σ is a nonempty minimal subset of A that is mapped onto itself by σ.

21. A *cycle* is a permutation having only one orbit.

22. The *alternating group* is the group of all even permutations.



9.22 Figure

23. Mark each of the following true or false.

_____ **a.** Every permutation is a cycle.

_____ **b.** Every cycle is a permutation.

_____ **c.** The definition of even and odd permutations could have been given equally well before Theorem 9.15.

_____ **d.** Every nontrivial subgroup H of S_9 containing some odd permutation contains a transposition.

_____ **e.** A_5 has 120 elements.

_____ **f.** S_n is not cyclic for any $n \geq 1$.

_____ **g.** A_3 is a commutative group.

_____ **h.** S_7 is isomorphic to the subgroup of all those elements of S_8 that leave the number 8 fixed.

_____ **i.** S_7 is isomorphic to the subgroup of all those elements of S_8 that leave the number 5 fixed.

_____ **j.** The odd permutations in S_8 form a subgroup of S_8.

24. Which of the permutations in S_3 of Example 8.7 are even permutations? Give the table for the alternating group A_3.

Proof Synopsis

25. Give a one-sentence synopsis of Proof 1 of Theorem 9.15.

26. Give a two-sentence synopsis of Proof 2 of Theorem 9.15.

Theory

27. Prove the following about S_n if $n \geq 3$.

a. Every permutation in S_n can be written as a product of at most $n - 1$ transpositions.

b. Every permutation in S_n that is not a cycle can be written as a product of at most $n - 2$ transpositions.

c. Every odd permutation in S_n can be written as a product of $2n + 3$ transpositions, and every even permutation as a product of $2n + 8$ transpositions.

28. a. Draw a figure like Fig. 9.16 to illustrate that if i and j are in different orbits of σ and $\sigma(i) = i$, then the number of orbits of $(i, j)\sigma$ is one less than the number of orbits of σ.
    b. Repeat part (a) if $\sigma(j) = j$ also.

29. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

30. Let σ be a permutation of a set A. We shall say "σ **moves** $a \in A$" if $\sigma(a) \neq a$. If A is a finite set, how many elements are moved by a cycle $\sigma \in S_A$ of length n?

31. Let A be an infinite set. Let H be the set of all $\sigma \in S_A$ such that the number of elements moved by σ (see Exercise 30) is finite. Show that H is a subgroup of S_n.

32. Let A be an infinite set. Let K be the set of all $\sigma \in S_A$ that move (see Exercise 30) at most 50 elements of A. Is K a subgroup of S_A? Why?

33. Consider S_n for a fixed $n \geq 2$ and let σ be a fixed odd permutation. Show that every odd permutation in S_n is a product of σ and some permutation in A_n.

34. Show that if σ is a cycle of odd length, then σ^2 is a cycle.

35. Following the line of thought opened by Exercise 34, complete the following with a condition involving n and r so that the resulting statement is a theorem:

    If σ is a cycle of length n, then σ^r is also a cycle if and only if...

36. Let G be a group and let a be a fixed element of G. Show that the map $\lambda_a : G \to G$, given by $\lambda_a(g) = ag$ for $g \in G$, is a permutation of the set G.

37. Referring to Exercise 36, show that $H = \{\lambda_a \mid a \in G\}$ is a subgroup of S_G, the group of all permutations of G.

38. Referring to Exercise 49 of Section 8, show that H of Exercise 37 is transitive on the set G. [*Hint:* This is an immediate corollary of one of the theorems in Section 4.]

39. Show that S_n is generated by $\{(1,2), (1,2,3,\cdots,n)\}$. [*Hint:* Show that as r varies, $(1,2,3,\cdots,n)^r(1,2)(1,2,3,\cdots,n)^{n-r}$ gives all the transpositions $(1,2), (2,3), (3,4), \cdots, (n-1,n), (n,1)$. Then show that any transposition is a product of some of these transpositions and use Corollary 9.12]

SECTION 10 Cosets and the Theorem of Lagrange

You may have noticed that the order of a subgroup H of a finite group G seems always to be a divisor of the order of G. This is the theorem of Lagrange. We shall prove it by exhibiting a partition of G into cells, all having the same size as H. Thus if there are r such cells, we will have

$$r(\text{order of } H) = (\text{order of } G)$$

from which the theorem follows immediately. The cells in the partition will be called *cosets of H*, and they are important in their own right. In Section 14, we will see that if H satisfies a certain property, then each coset can be regarded as an element of a group in a very natural way. We give some indication of such *coset groups* in this section to help you develop a feel for the topic.

Cosets

Let H be a subgroup of a group G, which may be of finite or infinite order. We exhibit two partitions of G by defining two equivalence relations, $\sim_L$ and $\sim_R$ on G.