Problem 1.

Give the definitions of any **four** of the following terms.

English version

i. A metric space
ii. A Cauchy sequence in a metric space
iii. A continuous map from one metric space to another
iv. The kernel (null space) of a linear transformation
v. An inner product on a complex vector space
vi. A Hilbert space
vii. A normed vector space

Solution:

i. A metric space is a pair (M, d) where M is a set and $d\colon M \times M \to [0, \infty)$ is a function. The function d, called the metric, must satisfy the following three conditions for all $x, y, z \in M$:
   (a) $d(x, y) = 0$ if and only if $x = y$,
   (b) $d(x, y) = d(y, x)$,
   (c) $d(x, z) \leq d(x, y) + d(y, z)$.

ii. A Cauchy sequence in a metric space, say (M, d), is a sequence in M, say (s_n), such that for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for $n, m > N$, $d(s_n, s_m) < \epsilon$.

iii. Let (M_1, d_1) and (M_2, d_2) be metric spaces. A continuous map from (M_1, d_1) to (M_2, d_2) is a function $f\colon M_1 \to M_2$ on the underlying sets which satisfies one of the following two equivalent conditions:
   (a) For every $x \in M_1$ and $\epsilon > 0$, there is a $\delta > 0$ such that whenever $y \in M_1$ is such that $d_1(x, y) < \delta$ then $d_2(f(x), f(y)) < \epsilon$.
   (b) The preimage under f of an open set in (M_2, d_2) is open in (M_1, d_1).

iv. The kernel (null space) of a linear transformation is the set of vectors in the domain which are mapped by the linear transformation to the zero vector in the codomain.

v. An inner product on a complex vector space, say V, is a map $V \times V \to \mathbb{C}$, written (u, v), satisfying the following conditions:
   (a) $(v, v) \geq 0$ for all $v \in V$ with equality if and only if $v = 0$;
   (b) $(\cdot, \cdot)$ is linear in the first argument; that is, $(u + \lambda v, w) = (u, w) + \lambda\,(v, w)$ for all $u, v, w \in V$ and $\lambda \in \mathbb{C}$;
   (c) $(u, v) = \overline{(v, u)}$.

vi. A Hilbert space is an inner product space that is complete for the metric induced by the inner product.

vii. A normed vector space is a pair $(X, \|\cdot\|)$ where X is a vector space and $\|\cdot\|$ is a norm on X. That is, $\|\cdot\|$ is a function $X \to [0, \infty)$ satisfying the following conditions:
   (a) $\|x\| = 0$ if and only if $x = 0$,
   (b) $\|\lambda x\| = |\lambda|\,\|x\|$ for all $x \in X$ and scalars λ,
   (c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Problem 2.

In this problem we equip $\mathbb{R}^3$ and $\mathbb{R}^4$ with their standard Euclidean norms:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \qquad (n = 3 \text{ or } 4)$$

Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 & -1 & 2 \\ 4 & -1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ -4 \\ -5 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -9 \end{bmatrix}$$

a. Find an $x \in \mathbb{R}^3$ such that $Ax = b$.
Solution:
Either by using Gaussian elimination, or simply by inspection, we see that

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

will do. There are more solutions, anything of the form

$$\begin{bmatrix} -1+t \\ 1+t \\ -t \end{bmatrix}$$

for $t \in \mathbb{R}$ will also do.

b. Find the set of $y \in \mathbb{R}^3$ such that Ay is the nearest point to c.
Solution:
We do this using least squares: solving $A^TAy = A^Tc$. Computing this yields

$$\begin{bmatrix} 30 & 0 & 30 \\ 0 & 15 & 15 \\ 30 & 15 & 45 \end{bmatrix} y = \begin{bmatrix} -30 \\ 15 \\ -15 \end{bmatrix}$$

Gaussian elimination (or inspection) reveals that the solution space of this is the same as the above:

$$\left\{ \begin{bmatrix} -1+t \\ 1+t \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

c. Find the point $z \in \mathbb{R}^3$ with the smallest norm such that Az is the nearest point to c.
Solution:
We compute the norm of the vectors in the solution space. For simplicity, we actually compute the square of the norm, since if the norm is minimal so is its square. We obtain:

$$\left\| \begin{bmatrix} -1+t \\ 1+t \\ -t \end{bmatrix} \right\| = (-1+t)^2 + (1+t)^2 + t^2 = 2 + 3t^2.$$

This is clearly minimal when $t = 0$ which corresponds to the vector

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Problem 3.

Let ℓ^∞ be the space of all bounded sequences in $\mathbb{R}$ with norm

$$\|(x_n)\|_\infty = \sup\{|x_n|\}.$$

Let $\ell^0 \subseteq \ell^\infty$ be the space of all sequences that are eventually zero; that is, $x = (x_n) \in \ell^0$ if there is some N (depending on x) such that $x_n = 0$ for $n \geq N$. Let $c_0 \subseteq \ell^\infty$ be the space of all sequences $x = (x_n)$ for which $\lim_{n\to\infty} x_n$ exists and is equal to zero.

Prove that c_0 is the closure of ℓ^0 in ℓ^∞.

Solution:

To prove that c_0 is the closure of ℓ^0 in ℓ^∞ we need to prove several things. First, that c_0 contains ℓ^0. Second, that c_0 is closed. Thirdly, that c_0 is the minimal space with those properties.

The first is straightforward. Let $(x_n) \in \ell^0$. Then there is some $N \in \mathbb{N}$ such that $x_n = 0$ for $n > N$. Thus for any $\epsilon > 0$, $|x_n - 0| = 0 < \epsilon$ for $n > N$. Hence $\lim_{n\to\infty}(x_n)$ exists and is zero. Thus $(x_n) \in c_0$ and so, as (x_n) was an arbitrary vector in ℓ^0, $\ell^0 \subseteq c_0$.

The second is perhaps the trickiest. There are two possible approaches: one is to show that c_0 contains all of its accumulation points; this other is to show that its complement is open.

Let us try the first method. We need to show that if (x^n) is a sequence of points in c_0 which converges in ℓ^∞, say to x, then $x \in c_0$. That is, we need to show that if we write $x = (x_n)$ then (x_n) has a limit and that limit is 0. Let $\epsilon > 0$. As $(x^n) \to x$, there is some $N \in \mathbb{N}$ such that for $n > N$, $\|x^n - x\|_\infty < \epsilon/2$. Fix $n > N$. As $x^n \in c_0$, (x^n_m) has a limit and that limit is 0 (here n is fixed and the sequence is taken over m). Thus there is some $N' \in \mathbb{N}$ such that for $m > N'$, $|x^n_m| < \epsilon/2$. Then for $m > N'$,

$$|x_m| \leq \left|x_m - x^n_m + x^n_m\right| \leq \left|x_m - x^n_m\right| + \left|x^n_m\right| \leq \|x - x^n\|_\infty + \left|x^n_m\right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $x \in c_0$.

Let us also show that c_0 is closed by the second method. For $x \in \ell^\infty$ that is *not* in c_0 we need to find an $\epsilon > 0$ such that if $\left\|x - y\right\|_\infty < \epsilon$ then $y \notin c_0$. So let $x \in \ell^\infty$ be such that $x \notin c_0$. As x is not in c_0, it does not converge to 0. So there is some $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is some $n > N$ with $|x_n| \geq 2\epsilon$. Let $y \in \ell^\infty$ be such that $\left\|x - y\right\|_\infty < \epsilon$. Then let $N \in \mathbb{N}$, and let $n > N$ be such that $|x_n| \geq 2\epsilon$. Then

$$\left|y_n\right| = \left|y_n - x_n + x_n\right| \geq \left|\left|y_n - x_n\right| - |x_n|\right|.$$

But $|x_n| \geq 2\epsilon$ and $|y_n - x_n| \leq \left\|x - y\right\|_\infty < \epsilon$ so $||y_n - x_n| - |x_n|| \geq \epsilon$. Hence y, as a sequence, does not converge to 0 and so $y \notin c_0$.

Finally, we need to show that if W is a closed set containing ℓ^0 then W contains c_0. To do this it is sufficient to show that each element of c_0 is the limit of some sequence in ℓ^0. If we can show this, then for any $x \in c_0$ there is some sequence $(x^n) \subseteq \ell^0$ converging to x. Then $(x^n) \subseteq W$ also so, since W is closed, $x \in W$. As x was an arbitrary element of c_0 this shows that $c_0 \subseteq W$.

So let us prove this final step. Let $x \in c_0$. For $n \in \mathbb{N}$, let $x^n \in \ell^0$ be the element obtained by truncating x at the nth step. That is, x^n agrees with x up to the nth component and is thereafter 0. We need to show that (x^n) converges to x. Let us write x^n_m for the mth component of x^n and x_m for the mth component of x. Let $\epsilon > 0$. As $x \in c_0$, there is some N such that for $n > N$, $|x_n| < \epsilon/2$. Now for $n > N$, $x - x^n$ is zero up to the nth component and is x_m for $m > n$. Thus

$$\|x - x^n\|_\infty = \sup\{|x_m| : m > n\}.$$

Since $n > N$, $|x_m| < \epsilon/2$ for all $m > n$. Hence $\sup\{|x_m| : m > n\} \leq \epsilon/2 < \epsilon$. Thus whenever $n > N$, $\|x - x^n\|_\infty < \epsilon$ and so (x^n) converges to x, as required.

Problem 4.

a. Let (X, d) be a complete metric space, $T\colon X \to X$ a contraction. Let $\alpha < 1$ be such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.
Let x^* be the fixed point of T. Let $x_0 \in X$ and recursively define $x_n = T(x_{n-1})$ for $n \geq 1$. Prove that
$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1) \qquad \text{for all } n \geq 1.$$
Solution:
There are a couple of different ways to prove this, but all use induction at one step.
Let us prove by induction that
$$d(x_n, x^*) \leq \alpha^n d(x_0, x^*).$$
For $n = 0$ we get a tautology: $d(x_0, x^*) \leq d(x_0, x^*)$. So assume that this holds for all $n \in \mathbb{N}$ and we shall prove that it holds for $n+1$. As x^* is the fixed point of T, $T(x^*) = x^*$. Also, by definition, $x_{n+1} = T(x_n)$. Hence
$$d(x_{n+1}, x^*) = d(T(x_n), T(x^*)) \leq \alpha d(x_n, x^*) \leq \alpha \cdot \alpha^n d(x_0, x^*) = \alpha^{n+1} d(x_0, x^*).$$
The first inequality comes from the fact that T is a contraction, the second from the inductive hypothesis.
Hence, by induction, $d(x_n, x^*) \leq \alpha^n d(x_0, x^*)$ for all n.
Now let us consider $d(x_0, x^*)$. By the triangle inequality we have
$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*).$$
Then $d(x_1, x^*) = d(T(x_0), T(x^*)) \leq \alpha d(x_0, x^*)$ as above. Rearranging gives $d(x_0, x^*) \leq (1-\alpha)^{-1} d(x_0, x_1)$ as required (note that $\alpha \neq 1$ so we can divide by $(1-\alpha)$. Putting this together with the previous part yields the required result.

b. Let $C([0,1], \mathbb{R})$ denote the vector space of all continuous functions from $[0,1]$ to $\mathbb{R}$. Define a map $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by
$$(Tx)(t) = 1 + \int_0^t sx(s)ds.$$
Show that T is a contraction and that one choice for α is $\frac{1}{2}$. Here, $C([0,1], \mathbb{R})$ is equipped with its standard metric: $d_\infty(f, g) = \max\{|f(t) - g(t)| : 0 \leq t \leq 1\}$.
Solution:
Let $x, y \in C([0,1], \mathbb{R})$ and consider $d_\infty(T(x), T(y))$. We want to show that this is less than $\frac{1}{2} d_\infty(x, y)$.
By definition,
$$d_\infty(T(x), T(y)) = \sup\{\left|(Tx)(t) - (Ty)(t)\right| : t \in [0,1]\}.$$
Thus we consider
$$(Tx)(t) - (Ty)(t) = 1 + \int_0^t sx(s)ds - 1 - \int_0^t sy(s)ds = \int_0^t s(x(s) - y(s))ds.$$
Now
$$\left|(Tx)(t) - (Ty)(t)\right| \leq \int_0^t s\left|x(s) - y(s)\right|ds \leq \int_0^t s d_\infty(x, y) ds = \int_0^t sds d_\infty(x, y) = \frac{1}{2} t^2 d_\infty(x, y).$$
Thus taking the supremum over $0 \leq t \leq 1$ we find that
$$d_\infty(T(x), T(y)) \leq \frac{1}{2} d_\infty(x, y).$$
Thus T is a contration and one can take $\frac{1}{2}$ as the constant of contraction.

c. The sequence produced starting with $x_0(t) = 0$ begins

$$
\begin{aligned}
x_1(t) &= 1 \\
x_2(t) &= 1 + t^2/2 \\
x_3(t) &= 1 + t^2/2 + t^4/8 \\
x_4(t) &= 1 + t^2/2 + t^4/8 + t^6/48
\end{aligned}
$$

with general term

$$x_n(t) = \sum_{k=0}^{n-1} \frac{t^{2k}}{2^k k!}$$

Let us write x^* for the fixed point of T. Using part a, estimate how many terms are needed to ensure that $\sum_{k=0}^{n-1} 1/2^k k!$ is within 0.001 of $x^*(1)$?

Solution:

From part a, we have the estimate for the error as

$$d_\infty(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d_\infty(x_0, x_1).$$

As d_∞ is a supremum, we thus see that

$$|x_n(1) - x^*(1)| \leq \frac{\alpha^n}{1-\alpha} d_\infty(x_0, x_1).$$

Now we know that we can take $\alpha = \frac{1}{2}$, and that x_0 is the constant function at 0 and x_1 at 1. Thus $d_\infty(x_0, x_1) = 1$ and we find that

$$|x_n(1) - x^*(1)| \leq \frac{1}{2^{n-1}}$$

Since we want to guarantee that $|x_n(1) - x^*(1)| < 0.001$ then we can do this if we choose n such that $\frac{1}{2^{n-1}} < 0.001$; equivalently, that $2^{n-1} > 1000$.

As anyone even vaguely famililar with computers ought to know, $2^{10} = 1024$.

Therefore $n - 1 = 10$; that is to say, $n = 11$.

d. To five significant figures, $x^*(1) = 1.6487$ whilst $x_5(1) = 1.6484$. What does this tell you about your previous answer?

Solution:

This does not actually tell us anything directly about our previous answer. Knowing that one particular term of a sequence is close to the limit does not give any information about when we can *guarantee* being close to the limit.

There is one circumstance in which knowing one value does give information about the rest. That is when the sequence is *monotonic*; either increasing or decreasing. In that case, each term of the sequence must be closer to the limit than the last. This is the case here (as the sequence formed by taking sums of positive numbers) so knowing that $x_5(1)$ is within the error is sufficient to say that we could have taken only 5 terms.

However, even if our previous answer was an overestimate, we were not asked to find the least such n and so our answer was still correct.

Note that this most certainly does not imply that there was anything wrong with taking α to be $\frac{1}{2}$, nor that we could have done better (we couldn't).

Problem 5.

Let $C([0,1],\mathbb{C})$ denote the space of continuous functions from $[0,1]$ to $\mathbb{C}$. Let

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt$$

be the standard inner product on $C([0,1],\mathbb{C})$.

a. Let $T\colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the linear transformation which sends $f \in C([0,1],\mathbb{C})$ to the function Tf given by $(Tf)(t) = tf(t)$. Show that T is self-adjoint; that is, that

$$(Tf,g) = (f,Tg)$$

for all $f,g \in C([0,1],\mathbb{C})$.

Solution:

As $[0,1] \subseteq \mathbb{R}$, $\overline{tf(t)} = t\overline{f(t)}$ for any $f \in C([0,1],\mathbb{C})$. Hence

$$\begin{aligned}(Tf,g) &= \int_0^1 (Tf)(t)g(t)dt = \int_0^1 tf(t)\overline{g(t)}dt \\ &= \int_0^1 f(t)t\,\overline{g(t)}dt = \int_0^1 f(t)\,\overline{tg(t)}dt = \int_0^1 f(t)\overline{(Tg)(t)}dt = (f,Tg)\end{aligned}$$

as required.

b. For $f,g \in C([0,1],\mathbb{C})$ define $(f,g)_T$ by

$$(f,g)_T = (Tf,g)\,.$$

Prove that $(\cdot,\cdot)_T$ is an inner product on $C([0,1],\mathbb{C})$.

Solution:

We need to prove the axioms of an inner product. Linearity in the first argument is simple:

$$(f+\lambda g,h)_T = (T(f+\lambda g),h) = (Tf+\lambda Tg,h) = (Tf,h) + \lambda\,(Tg,h) = (f,h)_T + \lambda\,(g,h)_T$$

where we have used the linearity of T and the linearity of $(\cdot,\cdot)$ in the first argument.

The conjugate symmetry follows from the previous result together with the fact that $(\cdot,\cdot)$ is known to be an inner product.

$$(f,g)_T = (Tf,g) = \overline{(g,Tf)} = \overline{(Tg,f)} = \overline{(g,f)_T}.$$

Finally, we need to prove that $(f,f)_T \geq 0$ with equality if and only if f is zero. There are a few ways to do this (though all are variations on the same theme). At one extreme, we notice that we can define an operator $S\colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ by $(Sf)(t) = t^{\frac{1}{2}}f(t)$ and this has the property that $S(Sf) = Tf$. This operator is also self-adjoint and so

$$(f,f)_T = (Tf,f) = (S(Sf),f) = (Sf,Sf)\,.$$

Since $(\cdot,\cdot)$ is an inner product, $(Sf,Sf) \geq 0$ with equality if and only if $Sf = 0$. Hence $(f,f)_T \geq 0$ with equality if and only if $Sf = 0$. But if $Sf = 0$ then $t^{\frac{1}{2}}f(t) = 0$ for all $t \in [0,1]$, whence $f(t) = 0$ for all $t \in (0,1]$ and so, as f is continuous, $f(t) = 0$ for all $t \in [0,1]$.

The more concrete approach is to observe that

$$(f,f)_T = \int_0^1 t\left|f(t)\right|^2 dt$$

and this is an integral of a positive continuous function, hence is positive and is zero if and only if the integrand is zero. Thus $(f,f)_T \geq 0$ with equality if and only if $t|f(t)|^2 = 0$ for all $t \in [0,1]$. This implies that $f(t) = 0$ for all $t \in (0,1]$, whence $f(t) = 0$ for all $t \in [0,1]$.

Either way, $(f,f)_T \geq 0$ with equality if and only if $f = 0$.

Hence $(\cdot,\cdot)_T$ is an inner product.

c. Let $V \subseteq C([0,1],\mathbb{C})$ be the subspace $\{a+bt+ce^t : a,b,c \in \mathbb{C}\}$. Find a, b such that

$$\int_0^1 t\left|e^t - a - bt\right|^2 dt$$

is minimal.

Solution:

This is a "least squares" question with space $V = \{a+bt+ce^t : a,b,c \in \mathbb{C}\}$ and inner product $(\cdot,\cdot)_T$. We are asked to find the closest point in the subspace $\{a+bt\}$ to the vector e^t. There are various inner products that will be useful. We compute:

$$\begin{aligned}
(1,1)_T &= \int_0^1 t\,dt = \frac{1}{2},\\
(t,1)_T &= \int_0^1 t^2 dt = \frac{1}{3},\\
(t,t)_T &= \int_0^1 t^3 dt = \frac{1}{4},\\
\left(e^t,1\right) &= \int_0^1 te^t dt = \left[te^t\right]_0^1 - \int_0^1 e^t dt = e - e + 1 = 1,\\
\left(e^t,t\right) &= \int_0^1 t^2 e^t dt = \left[t^2 e^t\right]_0^1 - 2\int_0^1 te^t dt = e - 2.
\end{aligned}$$

We see, therefore, that

$$1,\ t - \frac{(t,1)_T}{(1,1)_T}1$$

are orthogonal. Expanding, the second vector is $t - \frac{2}{3}$ which we multiply up to clear the denominator to get $3t-2$.

Now we compute

$$\begin{aligned}
(3t-2,3t-2)_T &= 9\,(t,t)_T - 12\,(t,1)_T + 4\,(1,1)_T = \frac{9}{4} - \frac{12}{3} + \frac{4}{2} = \frac{1}{4},\\
\left(e^t,3t-2\right)_T &= 3\left(e^t,t\right)_T - 2\left(e^t,1\right)_T = 3(e-2) - 2 = 3e - 8.
\end{aligned}$$

Hence the closest point to e^t is

$$\begin{aligned}
\frac{(e^t,3t-2)_T}{(3t-2,3t-2)_T}(3t-2) + \frac{(e^t,1)_T}{(1,1)_T}1 &= \frac{3e-8}{\frac{1}{4}}(3t-2) + \frac{1}{\frac{1}{2}}1\\
&= 4(3e-8)(3t-2) + 2 = (36e-96)t + (66-24e) \simeq 1.8581t + 0.76124.
\end{aligned}$$