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Suggested solutions
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Problem 1

For a given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon$ and $d(y_n, a) < \epsilon$ for all $n > N$. For the sequence $(z_n)_n$ we will then have for $k, l > 2N$ that

$$d(z_k, z_l) \leq d(z_k, a) + d(a, z_l) < \epsilon + \epsilon = 2\epsilon.$$

This shows that $(z_n)_n$ is a Cauchy sequence.

Problem 2

a) We see that rank $A = 2$, and we find the bases

$$\mathcal{R}(A) : \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \quad \mathcal{N}(A) : \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}, \quad \mathcal{R}(A^T) : \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

The basis for $\mathcal{N}(A^T)$ is spanned by the last row of L^{-1} (since $3 - 2 = 1$). We invert L,

$$[L \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] = [I \mid L^{-1}]$$

and find that

$$\mathcal{N}(A^T) : \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

b) We are given $B = [b_1|b_2|b_3]$. First we find the orthonormal vectors q_1, q_2 and q_3 using Gram-Schmidt.

$$\tilde{q}_1 = b_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{q}_2 = b_2 - \frac{\langle b_2, \tilde{q}_1 \rangle}{\langle \tilde{q}_1, \tilde{q}_1 \rangle} \tilde{q}_1 = b_2 + \frac{1}{2} \tilde{q}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}, \qquad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\tilde{q}_3 = b_3 - \frac{\langle b_3, \tilde{q}_1\rangle}{\langle \tilde{q}_1, \tilde{q}_1\rangle}\tilde{q}_1 - \frac{\langle b_3, \tilde{q}_2\rangle}{\langle \tilde{q}_2, \tilde{q}_2\rangle}\tilde{q}_2 = b_3 - 0\cdot\tilde{q}_1 + \tfrac{1}{3}\tilde{q}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix},$$

$$q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|} = \frac{1}{\sqrt{12}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

We have

$$\begin{aligned} b_1 &= \tilde{q}_1 &&= \sqrt{2}q_1 \\ b_2 &= -\tfrac{1}{2}\tilde{q}_1 + \tilde{q}_2 &&= -\tfrac{1}{2}\sqrt{2}q_1 + \tfrac{1}{2}\sqrt{6}q_2 \\ b_3 &= 0\cdot\tilde{q}_1 - \tfrac{2}{3}\tfrac{1}{6}\tilde{q}_2 + \tilde{q}_3 &&= 0\cdot q_1 - \tfrac{1}{3}\sqrt{6}q_2 + \tfrac{1}{3}\sqrt{12}q_3, \end{aligned}$$

and hence the QR-factorization of B is

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 0 & 0 & -3/\sqrt{12} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\sqrt{2}/2 & 0 \\ 0 & \sqrt{6}/2 & -\sqrt{6}/3 \\ 0 & 0 & \sqrt{12}/3 \end{bmatrix}.$$

c) First we calculate

$$C^TC = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

The characteristic polynomial of C^TC is $(2-\lambda)(2-\lambda) - 1 = 3 - 4\lambda + \lambda^2 = (3-\lambda)(1-\lambda)$, so 3 and 1 are eigenvalues, and $\sigma_1 = \sqrt{3}, \sigma_2 = 1$ are the singular values. Next, we find orthonormal eigenvectors for C^TC.

$$\lambda = 3: \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\lambda = 1: \quad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \quad\Rightarrow\quad q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We also want to find the columns of P. We have

$$p_1 = \frac{1}{\sigma_1}Cq_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix},$$

$$p_2 = \frac{1}{\sigma_2}Cq_2 = \frac{1}{1}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We see that

$$p_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is orthogonal to both p_1 and p_2 (we can also use Gram-Schmidt to find such a p_3), so a singular value decomposition of C is

$$C = P\Sigma Q^T = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Problem 3

a) Let $x, y \in E$. Then

i) $\|x\|_\infty > 0$ for $x \neq 0$ because of the absolute value,

ii) For $\lambda \in \mathbb{C}$, it is clear that $\|\lambda x\|_\infty = |\lambda|\|x\|_\infty$,

iii) For all $t \in [0,1]$ is

$$\big|x(t) + y(t)\big| \leq |x(t)| + |y(t)| \leq \|x\|_\infty + \|y\|_\infty.$$

Taking max on the left gives $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

Hence $\|\cdot\|_\infty$ is a norm on E.

Let $x, y, z \in E$ and $\lambda \in \mathbb{C}$. Then

i) - iii) By the properties of the integral, it is clear that $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ and $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

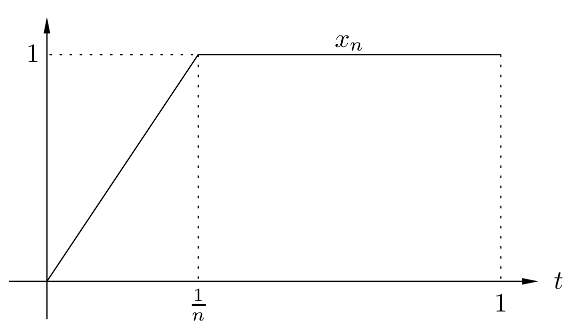
iv) $\langle x, x\rangle = \int_0^1 |x(t)|^2\, dt > 0$ and as x is continuous it follows that $\langle x, x\rangle > 0$ when $x \neq 0$.

So $\langle\cdot,\cdot\rangle$ is an inner product on E.

b) Let $(x_n)_n \subset M$ be a sequence that converges to $x \in E$ with respect to the metric d. In particular this implies that $x_n(0) \to x(0)$, and since $x_n(0) = 0$ for all n, we get that $x(0) = 0$ and $x \in M$. So M is closed with respect to d.

Let $x_n \in M$ be the function whose graph is given in the figure,

$$x_n(t) = \begin{cases} nt & \text{if } 0 \leq t \leq \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1 \end{cases}.$$



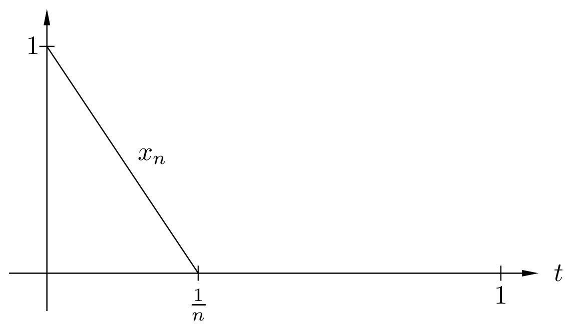
Let $\mathbf{1}$ be the function that is 1 for all t. Then

$$\tilde{d}(x_n, \mathbf{1}) = \left(\int_0^1 (x_n(t) - 1)^2\, dt\right)^{1/2} \leq \left(\int_0^{1/n} dt\right)^{1/2} = \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

So $x_n \to \mathbf{1}$ with respect to $\tilde{d}$, but $\mathbf{1} \notin M$, so M is not closed with respect to $\tilde{d}$.

c) It is sufficient (by Theorem 6.3 in Young) to consider the continuity properties of ϕ at 0. So let (x_n) be a sequence such that $x_n \to 0$ with respect to d. Then $x_n(0) \to 0$ and $\phi(x_n) = x_n(0) \to 0$. Hence ϕ is continuous at 0, and so ϕ is continuous with respect to d.
The functions x_n, whose graph is given in the figure, is defined to be

$$x_n(t) = \begin{cases} -nt + 1 & \text{if } 0 \le t \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le t \le 1 \end{cases}.$$



Then $x_n \to 0$ with respect to $\tilde{d}$, since

$$\langle x_n, x_n \rangle = \int_0^1 |x_n(t)|^2 \, dt \le \int_0^{1/n} \, dt = \frac{1}{n}.$$

However, $\phi(x_n) = x_n(0) = 1$, and so $\phi(x_n) \not\to 0$. So ϕ is discontinuous at 0, and so ϕ is discontinuous with respect to $\tilde{d}$.
(We could also have used that $\phi^{-1}(0) = M$ is not closed with respect to $\tilde{d}$.)

d) We use Gram-Schmidt to orthonormalize $\{1, t\}$ with respect to the given inner product $\langle \cdot, \cdot \rangle$. We then find that

$$e_1(t) = 1, \qquad e_2(t) = 2\sqrt{3}(t - \tfrac{1}{2})$$

are orthonormal and such that $\operatorname{span}\{1, t\} = \operatorname{span}\{e_1, e_2\} = U$. We get the integral to be minimal if $a + bt$ is chosen to be the orthogonal projection P of t^4 onto U.

$$\begin{aligned} P(t^4) &= \langle t^4, e_1 \rangle e_1 + \langle t^4, e_2 \rangle e_2 \\ &= \left(\int_0^1 t^4 \, dt \right) \cdot 1 + \left(2\sqrt{3} \int_0^1 t^4 (t - \tfrac{1}{2}) \, dt \right) 2\sqrt{3}(t - \tfrac{1}{2}) \\ &= -\tfrac{1}{5} + \tfrac{4}{5} t. \end{aligned}$$

Hence $a = -\frac{1}{5}$ and $b = \frac{4}{5}$.

Problem 4

a) Let $x \in X$ and $t \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\begin{aligned} |(Tx)(t)| &= \left| \int_0^t \left(x(\tau)^3 - 2\tau^2 \right) d\tau \right| \le \int_0^{|t|} \left(|x(\tau)|^3 + 2\tau^2 \right) d\tau \\ &\le \int_0^{|t|} \left(\tfrac{1}{8} + 2\tau^2 \right) d\tau = \tfrac{1}{8}|t| + \tfrac{2}{3}|t|^3 \le \tfrac{1}{16} + \tfrac{2}{3} \cdot \tfrac{1}{8} < \tfrac{1}{2}. \end{aligned}$$

Hence, $d(Tx, 0) \le \frac{1}{2}$, and $Tx \in X$.

b) Assume that $x, y \in X$. For $t \in [-\frac{1}{2}, \frac{1}{2}]$ we have

$$\begin{aligned}\big|(Tx)(t) - (Ty)(t)\big| &= \left|\int_0^t \big[f(\tau, x(\tau)) - f(\tau, y(\tau))\big]\, d\tau\right| \\ &\leq \int_0^{|t|} \big|f(\tau, x(\tau)) - f(\tau, y(\tau))\big|\, d\tau \leq \tfrac{3}{4}\int_0^{|t|} |x(\tau) - y(\tau)|\, d\tau \\ &\leq \tfrac{3}{4} d(x,y) \int_0^{|t|} d\tau \leq \tfrac{3}{8} d(x,y).\end{aligned}$$

Taking max on the left, we see that T is a contraction, with contraction constant $\alpha = \frac{3}{8}$.

c) The set X is a closed subset of the complete metric space $C[-\frac{1}{2}, \frac{1}{2}]$, and is thus complete. According to Banach's Fixed Point Theorem, will the sequence (x_n) converge to a (unique) fixed point $\tilde{x} \in C[-\frac{1}{2}, \frac{1}{2}]$. That is

$$\tilde{x}(t) = (T\tilde{x})(t), \quad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

or

$$\tilde{x}(t) = \int_0^t f(\tau, \tilde{x}(\tau))\, d\tau$$

Differentiate on both sides with respect to t to obtain

$$\tilde{x}'(t) = f(t, \tilde{x}(t)) = \tilde{x}(t)^3 - 2t^2, \quad t \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

In addition $\tilde{x}(0) = 0$.